Appendix B

Lemma 1. Let $\hat{\alpha}_1$ be the OLS estimator for α_1 in the two-level model in (23). If the true functional form relationship between potential outcomes and the treatment assignment score is correctly specified in the model, then, $\hat{\alpha}_1$ is a consistent estimator for α_1. Furthermore, as the number of units, *n*, increases to infinity in (23) and for fixed *m*, $\hat{\alpha}_1$ converges to a normal distribution with variance:

$$(B.2.1) \quad AsyVar_{RD}(\hat{\alpha}_1^{TL}) = \frac{1}{p(1-p)(1-\rho_{TS}^2)}\left[\frac{\sigma_\tau^2}{n} + \frac{\sigma_\delta^2}{nm}\right],$$

where ρ_{TS} is the correlation between T_i^{RD} and $Score_i$. A comparable expression can be obtained for the aggregated model in (8) by setting $\sigma_\delta^2 = 0$ and replacing σ_τ^2 with σ_η^2.

Proof. Write (23) in terms of centered random variables as follows:

$$(B.2.2) \quad w_{ij}^* = \alpha_1 T_i^* + \alpha_2 S_i^* + (\tau_i^* + \delta_{ij}^*),$$

where $w_{ij}^* = w_{ij}^{RD} - E(w_{ij}^{RD})$, $T_i^* = T_i^{RD} - p$, $S_i^* = Score_i - E(Score_i)$, $\tau_i^* = \tau_i - E(\tau_i)$ and $\delta_{ij}^* = \delta_{ij} - E(\delta_{ij})$. Let $\tilde{w}_{ij}$, $\tilde{T}_i$, $\tilde{S}_i$, $\tilde{\tau}_i$ and $\tilde{\delta}_{ij}$ be respective *empirically* centered variables. If $Z_i^* = (T_i^* \; S_i^*)$ and $\tilde{Z}_i = (\tilde{T}_i \; \tilde{S}_i)$, then the OLS estimator for the parameters in (B.2.2) is as follows:

$$\begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{pmatrix} = \left[\sum_{i=1}^{n}\sum_{j=1}^{m} \tilde{Z}_i'\tilde{Z}_i\right]^{-1}\left[\sum_{i=1}^{n}\sum_{j=1}^{m} \tilde{Z}_i'\tilde{w}_{ij}\right].$$

Standard asymptotic arguments can be used to prove that as *n* approaches infinity,

$$(B.2.3) \quad \begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{pmatrix} \xrightarrow{p} \left[mE(Z_i^{*\prime} Z_i^*)\right]^{-1} E(mZ_i^{*\prime} w_{ij}^*) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \left[mE(Z_i^{*\prime} Z_i^*)\right]^{-1} E\left[mZ_i^{*\prime}(\tau_i^* + \delta_{ij}^*)\right].$$

In this expression,

$$(B.2.4) \quad \left[mE(Z_i^{*\prime} Z_i^*)\right]^{-1} = \begin{bmatrix} mp(1-p) & m\sigma_{TS} \\ m\sigma_{TS} & m\sigma_S^2 \end{bmatrix}^{-1} = \begin{bmatrix} \sigma_S^2 & -\sigma_{TS} \\ -\sigma_{TS} & p(1-p) \end{bmatrix}\left(\frac{1}{m(\sigma_S^2 p(1-p) - \sigma_{TS}^2)}\right)$$

and

$$E(mZ_i^{*\prime}[\tau_i^* + \delta_{ij}^*]) = \begin{pmatrix} m\sigma_{T\tau} \\ m\sigma_{S\tau} \end{pmatrix},$$

where $\sigma_{T\tau}$ is the covariance between T_i^{RD} and τ_i, and $\sigma_{S\tau}$ is the covariance between *Score$_i$* and τ_i. Note that the covariance between Z_i^* and δ_{ij}^* is zero because T_i^{RD} and *Score$_i$* do not vary within schools. Thus, after some algebra, it can be seen that as *n* approaches infinity,

$$(B.2.5) \quad \hat{\alpha}_1 \xrightarrow{p} \alpha_1 + \left(\frac{\sigma_S^2 \sigma_{T\tau} - \sigma_{TS}\sigma_{S\tau}}{\sigma_S^2 p(1-p) - \sigma_{TS}^2} \right).$$

The second term on the right-hand-side of (B.2.5) is zero because it is the coefficient estimate on T_i^{RD} when τ_i is regressed on T_i^{RD} and *Score$_i$*. This conditional expectation is zero, because controlling for *Score$_i$*, there is no variation in treatment status. (Note that this result does not hold if the model is specified incorrectly and τ_i contains omitted score variables.) Thus, $\hat{\alpha}_1$ is asymptotically unbiased.

To obtain the asymptotic distribution of the two-level OLS estimator, we can rewrite (B.2.3) as follows:

$$\sqrt{n}\begin{pmatrix} \hat{\alpha}_1 - \alpha_1 \\ \hat{\alpha}_2 - \alpha_2 \end{pmatrix} = n^{-1/2}\left[mE(Z_i^{*\prime} Z_i^*) \right]^{-1} \sum_{i=1}^{n}\sum_{j=1}^{m} Z_i^{*\prime}(\tau_i^* + \delta_{ij}^*) + o_p(1),$$

where $o_p(1)$ denotes a term that converges in probability to zero. Thus, using (B.2.4), we find after some algebra that

$$(B.2.6) \quad \sqrt{n}(\hat{\alpha}_1 - \alpha_1) = n^{-1/2} \frac{1}{m\left(\sigma_S^2 p(1-p) - \sigma_{TS}^2\right)} \sum_{i=1}^{n} (m\tau_i^* + \sum_{j}^{m} \delta_{ij}^*)(\sigma_S^2 T_i^* - \sigma_{TS} S_i^*) + o_p(1).$$

Because $E\left[(m\tau_i^* + \sum_{j=1}^{m} \delta_{ij}^*)(\sigma_S^2 T_i^* - \sigma_{TS} S_i^*) \right] = 0$, a simple application of the central limit theorem (see, for example, Rao 1973) can be used to show that $\hat{\alpha}_1$ is asymptotically normally distributed with mean zero and the following variance:

$$(B.2.7)\ AsyVar(\hat{\alpha}_1) = \frac{E(m\tau_i^* + \sum_{j=1}^{m} \delta_{ij}^*)^2 E(\sigma_S^2 T_i^* - \sigma_{TS} S_i^*)^2}{nm^2[\sigma_S^2 p(1-p) - \sigma_{TS}^2]^2} = \frac{\sigma_S^2(\sigma_\tau^2 + [\sigma_\delta^2 / m])}{n[\sigma_S^2 p(1-p) - \sigma_{TS}^2]}.$$

The expressions in (B.2.7) and (B.2.1) are equivalent because $\sigma_{TS}^2 = \sigma_S^2 p(1-p)\rho_{TS}^2$.