Appendix A: Proof of Asymptotic Results for the ANCOVA Estimator

Lemma 1. Let $\hat{\gamma}_{ANCOVA}$ be the OLS estimator for γ in the two-level model in (13). Then, as the number of units, *n*, increases to infinity and for fixed *m*, $\hat{\gamma}_{ANCOVA}$ converges to a normal distribution with mean $\alpha_1 - \beta_1(\sigma_{01}/\sigma_0^2)$ and the following asymptotic variance:

$$(A.1) \quad AsyVar(\hat{\gamma}_{ANCOVA}) = \frac{1}{p(1-p)}\left[\frac{\sigma_1^2(1-\rho_{01}^2)}{n} + \frac{\tau_1^2(1-\lambda_{01}^2)}{nm}\right]\left[1+\frac{\beta_1^2 p(1-p)}{\sigma_0^2}\right]$$

Proof: It is convenient to express (6) and (7) in terms of centered random variables:

$$(A.2) \quad y_{1ij}^* = \alpha_1 T_i^* + (u_{1i}^* + e_{1ij}^*)$$
$$(A.3) \quad y_{0ij}^* = \beta_1 T_i^* + (u_{0i}^* + e_{0ij}^*),$$

where $T_i^* = T_i - p$, $y_{kij}^* = y_{kij} - E(y_{kij})$, $u_{ki}^* = u_{ki} - E(u_{ki})$ and $e_{kij}^* = e_{kij} - E(e_{kij})$ for $k = 0,1$. Let $\tilde{y}_{kij}$, $\tilde{T}_i$, $\tilde{u}_{ki}$ and $\tilde{e}_{kij}$ be respective *empirically* centered variables. Furthermore, let $\mathbf{X_{ij}} = [T_i \; \overline{y}_{0i} \; y_{0ij}^w]$, and $\mathbf{X}_{\mathbf{ij}}^*$ and $\tilde{\mathbf{X}}_{\mathbf{ij}}$ be associated centered row vectors of model covariates. Finally, let $\mathbf{y}_{\mathbf{1}}^*$ denote the vector of y_{1ij}^* values, $\mathbf{X}^*$ denote the matrix of X_{ij}^* values, and $\boldsymbol{\delta}' = (\gamma \; \delta_1 \; \delta_2)$ denote the parameter vector in (13).

As *n* approaches infinity (for fixed *m*) the OLS estimator $\hat{\boldsymbol{\delta}}$ in (13) converges to the following vector:

$$(A.4) \quad \hat{\boldsymbol{\delta}} = \left[\sum_{i=1}^{n}\sum_{j=1}^{m} \tilde{\mathbf{X}}_{\mathbf{ij}}^{/}\tilde{\mathbf{X}}_{\mathbf{ij}} / nm\right]^{-1}\left[\sum_{i=1}^{n}\sum_{j=1}^{m} \tilde{\mathbf{X}}_{\mathbf{ij}}^{/}\tilde{y}_{1ij} / nm\right] \xrightarrow{p} \left[E(\mathbf{X}_{\mathbf{ij}}^{*/}\mathbf{X}_{\mathbf{ij}}^*)\right]^{-1}\left[E(\mathbf{X}_{\mathbf{ij}}^{*/}y_{1ij}^*)\right].$$

By inserting (A.2) and (A.3) into (A.4), it can be shown that:

$$(A.5) \quad \left[E(\mathbf{X}_{\mathbf{ij}}^{*/}\mathbf{X}_{\mathbf{ij}}^*)\right]^{-1} = \frac{1}{p(1-p)\sigma_0^2\tau_0^2}\begin{bmatrix} [\sigma_0^2 + \beta_1^2 p(1-p)]\tau_0^2 & -\beta_1 p(1-p)\tau_0^2 & 0 \\ -\beta_1 p(1-p)\tau_0^2 & p(1-p)\tau_0^2 & 0 \\ 0 & 0 & p(1-p)\sigma_0^2 \end{bmatrix},$$

and

$$(A.6) \quad E(\mathbf{X}_{\mathbf{ij}}^{*/}y_{1ij}^*) = \begin{bmatrix} \alpha_1 p(1-p) \\ \alpha_1\beta_1 p(1-p) + \sigma_{01} \\ \lambda_{01} \end{bmatrix}.$$

It follows then that:

$$(A.7)\quad \hat{\gamma}_{ANCOVA} \xrightarrow{p} \gamma = \alpha_1 - \beta_1(\sigma_{01} / \sigma_0^2),$$
$$\hat{\delta}_1 \xrightarrow{p} \delta_1 = (\sigma_{01} / \sigma_0^2), \text{ and}$$
$$\hat{\delta}_2 \xrightarrow{p} \delta_2 = (\lambda_{01} / \tau_0^2).$$

To obtain the asymptotic distribution of the two-level OLS estimator, we rewrite the right-hand-side of (A.4) as follows:

$$(A.8)\quad \sqrt{nm}(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}) = \sqrt{nm}\left[\left[E(\mathbf{X}_{\mathbf{ij}}^{*/}\mathbf{X}_{\mathbf{ij}}^{*})\right]^{-1} \frac{\mathbf{X}^{*/}(\mathbf{y}_{\mathbf{1}}^{*} - \mathbf{X}^{*/}\boldsymbol{\delta})}{nm}\right] + o_p(1),$$

where $o_p(1)$ denotes a vector that converges in probability to zero. Because $E\left[\mathbf{X}^{*/}(\mathbf{y}_{\mathbf{1}}^{*} - \mathbf{X}^{*/}\boldsymbol{\delta})\right] = \mathbf{0}$, a simple application of the central limit theorem (see, for example, Rao 1973) can be used to show that $\hat{\boldsymbol{\delta}}$ is asymptotically normally distributed with the following variance:

$$(A.9)\quad AsyVar(\hat{\boldsymbol{\delta}}) = \left[E(\mathbf{X}_{\mathbf{ij}}^{*/}\mathbf{X}_{\mathbf{ij}}^{*})\right]^{-1} E[\mathbf{X}^{*/}(\mathbf{y}_{\mathbf{1}}^{*} - \mathbf{X}^{*/}\boldsymbol{\delta})(\mathbf{y}_{\mathbf{1}}^{*} - \mathbf{X}^{*/}\boldsymbol{\delta})'\mathbf{X}^{*}]\left[E(\mathbf{X}_{\mathbf{ij}}^{*/}\mathbf{X}_{\mathbf{ij}}^{*})\right]^{-1}.$$

The variance in (A.1) then follows using formulas from above and additional algebra to calculate the expectations in (A.9).