IES PRACTICE GUIDE WHAT WORKS CLEARINGHOUSE

Developing Effective Fractions Instruction for Kindergarten Through 8th Grade

NCEE 2010-4039
U.S. DEPARTMENT OF EDUCATION

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The goal of this practice guide is to offer educators specific evidence-based recommendations that address the challenge of improving students' understanding of fraction concepts in kindergarten through 8th grade. The guide provides practical, clear information on critical topics related to the teaching of fractions and is based on the best available evidence as judged by the authors.

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Improving Reading Comprehension in Kindergarten Through 3rd Grade *(September 2010)*		◆	
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Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools *(April 2009)*		◆	◆
Developing Effective Fractions Instruction for Kindergarten Through 8th Grade *(September 2010)*		◆	◆
Improving Adolescent Literacy: Effective Classroom and Intervention Practices *(August 2008)*		◆	◆
Structuring Out-of-School Time to Improve Academic Achievement *(July 2009)*		◆	◆
Dropout Prevention *(August 2008)*			◆
Helping Students Navigate the Path to College: What High Schools Can Do *(September 2009)*			◆

Developing Effective Fractions Instruction for Kindergarten Through 8th Grade

September 2010

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This report was prepared for the National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences under Contract ED-07-CO-0062 by the What Works Clearinghouse, which is operated by Mathematica Policy Research.

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Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L., & Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade: A practice guide* (NCEE #2010-4039). Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. Retrieved from whatworks.ed.gov/publications/practiceguides.

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Developing Effective Fractions Instruction for Kindergarten Through 8th Grade

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Review of Recommendations

Recommendation 1.

Build on students' informal understanding of sharing and proportionality to develop initial fraction concepts.

- Use equal-sharing activities to introduce the concept of fractions. Use sharing activities that involve dividing sets of objects as well as single whole objects.
- Extend equal-sharing activities to develop students' understanding of ordering and equivalence of fractions.
- Build on students' informal understanding to develop more advanced understanding of proportional reasoning concepts. Begin with activities that involve similar proportions, and progress to activities that involve ordering different proportions.

Recommendation 2.

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

- Use measurement activities and number lines to help students understand that fractions are numbers, with all the properties that numbers share.
- Provide opportunities for students to locate and compare fractions on number lines.
- Use number lines to improve students' understanding of fraction equivalence, fraction density (the concept that there are an infinite number of fractions between any two fractions), and negative fractions.
- Help students understand that fractions can be represented as common fractions, decimals, and percentages, and develop students' ability to translate among these forms.

Recommendation 3.

Help students understand why procedures for computations with fractions make sense.

- Use area models, number lines, and other visual representations to improve students' understanding of formal computational procedures.
- Provide opportunities for students to use estimation to predict or judge the reasonableness of answers to problems involving computation with fractions.
- Address common misconceptions regarding computational procedures with fractions.
- Present real-world contexts with plausible numbers for problems that involve computing with fractions.

Recommendation 4.

Develop students' conceptual understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems.

- Develop students' understanding of proportional relations before teaching computational procedures that are conceptually difficult to understand (e.g., cross-multiplication). Build on students' developing strategies for solving ratio, rate, and proportion problems.
- Encourage students to use visual representations to solve ratio, rate, and proportion problems.
- Provide opportunities for students to use and discuss alternative strategies for solving ratio, rate, and proportion problems.

Recommendation 5.

Professional development programs should place a high priority on improving teachers' understanding of fractions and of how to teach them.

- Build teachers' depth of understanding of fractions and computational procedures involving fractions.
- Prepare teachers to use varied pictorial and concrete representations of fractions and fraction operations.
- Develop teachers' ability to assess students' understandings and misunderstandings of fractions.

Acknowledgments

The panel greatly appreciates the efforts of Jeffrey Max, Moira McCullough, Andrew Gothro, and Sarah Prenovitz, staff from Mathematica Policy Research who participated in the panel meetings, summarized the research findings, and drafted the guide. Jeffrey Max and Moira McCullough had primary responsibility for drafting and revising the guide. We also thank Shannon Monahan, Cassandra Pickens, Scott Cody, Neil Seftor, Kristin Hallgren, and Alison Wellington for helpful feedback and reviews of earlier versions of the guide, and Laura Watson-Sarnoski and Joyce Hofstetter for formatting and producing the guide.

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Institute of Education Sciences Levels of Evidence for Practice Guides

This section provides information about the role of evidence in Institute of Education Sciences' (IES) What Works Clearinghouse (WWC) practice guides. It describes how practice guide panels determine the level of evidence for each recommendation and explains the criteria for each of the three levels of evidence (strong evidence, moderate evidence, and minimal evidence).

The level of evidence assigned to each recommendation in this practice guide represents the panel's judgment of the quality of the existing research to support a claim that when these practices were implemented in past research, positive effects were observed on student outcomes. After careful review of the studies supporting each recommendation, panelists determine the level of evidence for each recommendation using the criteria in Table 1 and the evidence heuristic depicted in Appendix E. The panel first considers the relevance of individual studies to the recommendation, and then discusses the entire evidence base, taking into consideration:

- the number of studies
- the quality of the studies
- whether the studies represent the range of participants and settings on which the recommendation is focused
- whether findings from the studies can be attributed to the recommended practice
- whether findings in the studies are consistently positive

A rating of *strong evidence* refers to consistent evidence that the recommended strategies, programs, or practices improve student outcomes for a wide population of students. In other words, there is strong causal and generalizable evidence.

A rating of *moderate evidence* refers either to evidence from studies that allow strong causal conclusions but cannot be generalized with assurance to the population on which a recommendation is focused (perhaps because the findings have not been widely replicated) or to evidence from studies that are generalizable but have some causal ambiguity. It also might be that the studies that exist do not specifically examine the outcomes of interest in the practice guide although they may be related.

A rating of *minimal evidence* suggests that the panel cannot point to a body of research that demonstrates the practice's positive effect on student achievement. In some cases, this simply means that the recommended practices would be difficult to study in a rigorous, experimental fashion;[1] in other cases, it means that researchers have not yet studied this practice, or that there is weak or conflicting evidence of effectiveness. A minimal evidence rating does not indicate that the recommendation is any less important than other recommendations with a strong evidence or moderate evidence rating.

Following WWC guidelines, improved outcomes are indicated by either a positive statistically significant effect or a positive substantively important effect size.[2] The WWC defines substantively important, or large, effects on outcomes to be those with effect sizes greater than 0.25 standard deviations. In this guide, the panel discusses substantively important findings as ones that contribute to the evidence of practices' effectiveness, even when those effects are not statistically significant.

Table 1. Institute of Education Sciences levels of evidence for practice guides

Strong Evidence

A rating of *strong evidence* means high-quality causal research links this practice with positive results in schools and classrooms. The research rules out other causes of the positive results, and the schools and classrooms are similar to those targeted by this guide. Strong evidence is demonstrated when an evidence base has the following properties:

- High internal validity: the evidence base consists of high-quality causal designs that meet WWC standards with or without reservations.[3]
- High external validity: the evidence base consists of a variety of studies with high internal validity that represent the population on which the recommendation is focused.[4]
- Consistent positive effects on relevant outcomes without contradictory evidence (i.e., no statistically significant negative effects) in studies with high internal validity.
- Direct relevance to scope (i.e., ecological validity), including relevant context (e.g., classroom vs. laboratory), sample (e.g., age and characteristics), and outcomes evaluated.
- Direct test of the recommendation in the studies or the recommendation is a major component of the interventions evaluated in the studies.
- The panel has a high degree of confidence that this practice is effective.
- In the particular case of recommendations on assessments, the evidence base meets *The Standards for Educational and Psychological Testing* (American Educational Research Association, American Psychological Association, and National Council on Measurement in Education, 1999).

Moderate Evidence

A rating of *moderate evidence* means high-quality causal research links this practice with positive results in schools and classrooms. However, the research may not adequately rule out other causes of the positive results, or the schools and classrooms are not similar to those targeted by this guide. Moderate evidence is demonstrated when an evidence base has the following properties:

- High internal validity but moderate external validity (i.e., studies that support strong causal conclusions, but generalization is uncertain) OR studies with high external validity but moderate internal validity (i.e., studies that support the generality of a relation, but the causality is uncertain).
  - The research may include studies meeting WWC standards with or without reservations with small sample sizes and/or other conditions of implementation or analysis that limit generalizability.
  - The research may include studies that support the generality of a relation but do not meet WWC standards;[5] however, they have no major flaws related to internal validity other than lack of demonstrated equivalence at pretest for quasi-experimental design studies (QEDs). QEDs without equivalence must include a pretest covariate as a statistical control for selection bias. These studies must be accompanied by at least one relevant study meeting WWC standards with or without reservations.
- A preponderance of positive effects on relevant outcomes. Contradictory evidence (i.e., statistically significant negative effects) must be discussed by the panel and considered with regard to relevance to the scope of the guide and intensity of the recommendation as a component of the intervention evaluated. If outcomes are out of the scope of the guide, this also must be discussed.
- The panel determined that the research does not rise to the level of strong evidence but is more compelling than a minimal level of evidence.
- In the particular case of recommendations on assessments, there must be evidence of reliability that meets *The Standards for Educational and Psychological Testing*, but evidence of validity may be from samples not adequately representative of the population on which the recommendation is focused.

(continued)

Table 1. Institute of Education Sciences levels of evidence for practice guides *(continued)*

Minimal Evidence
A rating of *minimal evidence* means the panel concluded the recommended practice should be adopted; however, the panel cannot point to a body of causal research that demonstrates the recommendation's positive effect and that rises to the level of moderate or strong evidence.

In terms of the levels of evidence indicated in Table 1, the panel relied on WWC evidence standards to assess the quality of evidence supporting educational programs and practices. WWC evaluates evidence for the causal validity of instructional programs and practices according to WWC standards.

Information about these standards is available at http://ies.ed.gov/ncee/wwc/pdf/wwc_procedures_v2_standards_handbook.pdf. Eligible studies that meet WWC evidence standards or meet evidence standards with reservations are indicated by **bold text** in the endnotes and references pages.

Introduction to the *Developing Effective Fractions Instruction for Kindergarten Through 8th Grade* Practice Guide

This section provides an overview of the importance of developing effective fractions instruction for kindergarten through 8th grade and explains key parameters considered by the panel in developing the practice guide. It also summarizes the recommendations for readers and concludes with a discussion of the research supporting the practice guide.

U.S. students' mathematics skills have fallen short for many years, with the ramifications of this inadequate knowledge widely recognized.

The 1983 report *A Nation at Risk* related America's safety and prosperity to its mathematical competence and warned that American students' mathematical knowledge was insufficient to meet the challenges of the modern world. More than 25 years later, U.S. students' mathematical achievement continues to lag far behind that of students in East Asia and much of Europe.[6] Only a small percentage of U.S. students possess the mathematics knowledge needed to pursue careers in science, technology, engineering, or mathematics (STEM) fields.[7] Many high school graduates lack the mathematical competence for a wide range of well-paying jobs in today's economy.[8] Moreover, large gaps in mathematics knowledge exist among students from different socioeconomic backgrounds and racial and ethnic groups within the United States.[9] These disparities hurt the national economy and also limit tens of millions of Americans' occupational and financial opportunities.[10]

Poor understanding of fractions is a critical aspect of this inadequate mathematics knowledge. Knowledge of fractions differs even more between students in the United States and students in East Asia than does knowledge of whole numbers.[11] This learning gap is especially problematic because understanding fractions is essential for algebra and other more advanced areas of mathematics.[12]

Teachers are aware of students' difficulty in learning about fractions and often are frustrated by it. In a recent national poll, Algebra I teachers rated their students as having "very poor preparation in rational numbers and operations involving fractions and decimals."[13] The algebra teachers ranked poor understanding of fractions as one of the two most important weaknesses in students' preparation for their course.

Many examples illustrate American students' weak understanding of fractions. On the 2004 National Assessment of Educational Progress (NAEP), 50% of 8th-graders could not order three fractions from least to greatest.[14] The problem is not limited to rational numbers written in common fraction notation. On the 2004 NAEP, fewer than 30% of 17-year-olds correctly translated 0.029 as $^{29}/_{1000}$.[15] The same difficulty is apparent in one-on-one testing of students in controlled experimental settings: when asked which of two decimals, 0.274 and 0.83, is greater, most 5th- and 6th-graders choose 0.274.[16]

These examples and others led the authors of this guide to conclude the following:

> A high percentage of U.S. students lack conceptual understanding of fractions, even after studying fractions for several years; this, in turn, limits students' ability to solve problems with fractions and to learn and apply computational procedures involving fractions.

The lack of conceptual understanding has several facets, including

- Not viewing fractions as numbers at all, but rather as meaningless symbols that need to be manipulated in arbitrary ways to produce answers that satisfy a teacher.

- Focusing on numerators and denominators as separate numbers rather than thinking of the fraction as a single number. Errors such as believing that $\frac{3}{8} > \frac{3}{5}$ arise from comparing the two denominators and ignoring the essential relation between each fraction's numerator and its denominator.
- Confusing properties of fractions with those of whole numbers. This is evident in many high school students' claim that just as there is no whole number between 5 and 6, there is no number of any type between $\frac{5}{7}$ and $\frac{6}{7}$.[17]

This practice guide presents five recommendations intended to help educators improve students' understanding of, and problem-solving success with, fractions. Recommendations progress from proposals for how to build rudimentary understanding of fractions in young children; to ideas for helping older children understand the meaning of fractions and computations that involve fractions; to proposals intended to help students apply their understanding of fractions to solve problems involving ratios, rates, and proportions. Improving students' learning about fractions will require teachers' mastery of the subject and their ability to help students master it; therefore, a recommendation regarding teacher education also is included.

Recommendations in the practice guide were developed by a panel of eight researchers and practitioners who have expertise in different aspects of the topic. Panelists include a mathematician active in issues related to mathematics teacher education; three mathematics educators, one of whom has been president of the National Council of Teachers of Mathematics; two psychologists whose research focuses on how children learn mathematics; and two practitioners who have taught mathematics in elementary and middle school classrooms and supervised other elementary and middle school mathematics teachers. Panel members worked collaboratively to develop recommendations based on the best available research evidence and on their combined experience and expertise regarding mathematics teaching and learning.

Scope of the practice guide

Writing this guide required decisions regarding the intended audience, which grade levels to examine, which skills and knowledge to consider, and which terms to use in describing the research and recommendations. The panel consistently chose to make the guide as inclusive as possible.

Audience and grade level. The intended audience is elementary and middle school teachers, mathematics supervisors, teacher leaders, specialists, coaches, principals, parents, teacher educators, and others interested in improving students' mathematics learning. Grade levels emphasized are kindergarten through 8th grade; almost all instruction in fractions takes place within this period, and this is the population studied in most of the available research. The guide focuses not only on computation with fractions, but also on skills that reflect understanding of fractions, such as estimating fractions' positions on number lines and comparing the sizes of fractions, because lack of such understanding underlies many of the other difficulties students have with fractions.

Content. This document uses the term *fractions* rather than *rational numbers*. The term *fractions* refers to the full range of ways of expressing rational numbers, including decimals, percentages, and negative fractions. The panel makes recommendations on this full range of rational numbers because students' understanding of them is critical to their use of fractions in context.

> The guide's inclusiveness is further evident in its emphasis on the need for students to be able to perform computational operations with fractions; to understand these computational operations; and to understand, more broadly, what fractions represent.

To help students understand the full range of fractions, the panel suggests educators effectively convey the following:

- Common fractions, decimals, and percents are equivalent ways of expressing the same number ($42/100 = 0.42 = 42\%$).
- Whole numbers are a subset of rational numbers.
- Any fraction can be expressed in an infinite number of equivalent ways ($3/4 = 6/8 = 9/12 = 0.75 = 75\%$, and so on).

Both the strengths students bring to the task of learning about fractions and the challenges that often make learning difficult are covered in this guide. Children enter school with a rudimentary understanding of sharing and proportionality, concepts on which teachers can build to produce more advanced understandings of fractions.[18] The scope of the guide includes describing these early developing concepts and how more advanced understanding can be built on them. The guide also describes common misconceptions about fractions that interfere with students' learning—for example, the misconception that multiplying two numbers must result in a larger number—and how such misconceptions can be overcome.

Finally, the guide addresses not only the need to improve students' understanding of fractions, but also the need to improve teachers' understanding of them. Far too many U.S. teachers can apply standard computational algorithms to solve problems involving fractions but do not know why those algorithms work or how to evaluate and explain why alternative procedures that their students generate are correct or incorrect.[19] Similarly, many teachers can explain part-whole interpretations of fractions but not other essential interpretations, such as considering fractions as measures of quantities that offer precision beyond that offered by whole numbers or viewing fractions as quotients.

U.S. teachers' understanding of fractions lags far behind that of teachers in nations that produce better student learning of fractions, such as Japan and China.[20] Although some of the information in this guide is aimed at deepening teachers' understanding of fractions, professional development activities that improve teachers' understanding of fractions and computational procedures that involve fractions also seem essential.

Summary of the recommendations

This practice guide includes five recommendations for improving students' learning of fractions. The first recommendation is aimed at building the foundational knowledge of young students, the next three target older students as they advance through their elementary and middle school years, and the final recommendation focuses on increasing teachers' ability to help students understand fractions. **Although the recommendations vary in their particulars, all five reflect the perspective that conceptual understanding of fractions is essential** for students to learn about the topic, to remember what they learned, and to apply this knowledge to solve problems involving fractions. Educators may profitably adopt some of the recommendations without adopting all of them, but we believe that the greatest benefit will come from adopting all of the recommendations that are relevant to their classes.

- **Recommendation 1** is to build on students' informal understanding of sharing and proportionality to develop initial fraction concepts. Learning is often most effective when it builds on existing knowledge, and fractions are no exception. By the time children begin school, most have developed a basic understanding of sharing that allows them to divide a region or set of objects equally among two or more people. These sharing activities can be used to illustrate concepts such as halves, thirds, and fourths, as well as more general concepts relevant to fractions, such

as that increasing the number of people among whom an object is divided results in a smaller fraction of the object for each person. Similarly, early understanding of proportions can help kindergartners compare, for example, how one-third of the areas of a square, rectangle, and circle differ.

- **Recommendation 2** is to ensure that students know that fractions are numbers that expand the number system beyond whole numbers, and to use number lines as a key representational tool to convey this and other fraction concepts from the early grades onward. Although it seems obvious to most adults that fractions are numbers, many students in middle school and beyond cannot identify which of two fractions is greater, indicating that they have cursory knowledge at best. Number lines are particularly advantageous for assessing knowledge of fractions and for teaching students about them. They provide a common tool for representing the sizes of common fractions, decimals, and percents; positive and negative fractions; fractions that are less than one and greater than one; and equivalent and nonequivalent fractions. Number lines also are a natural way of introducing students to the idea of fractions as measures of quantity, an important idea that needs to be given greater emphasis in many U.S. classrooms.

- **Recommendation 3** is to help students understand why procedures for computations with fractions make sense. Many U.S. students, and even teachers, cannot explain why common denominators are necessary to add and subtract fractions but not to multiply and divide them. Few can explain the "invert and multiply rule," or why dividing by a fraction can result in a quotient larger than the number being divided. Students sometimes learn computational procedures by rote, but they also often quickly forget or become confused by these routines; this is what tends to happen with fractions algorithms. Forgetting and confusing algorithms occur less often when students understand how and why computational procedures yield correct answers.

- **Recommendation 4** involves focusing on problems involving ratios, rates, and proportions. These applications of fraction concepts often prove difficult for students. Illustrating how diagrams and other visual representations can be used to solve ratio, rate, and proportion problems and teaching students to use them are important for learning algebra. Also useful is providing instruction on how to translate statements in word problems into mathematical expressions involving ratio, rate, and proportion. These topics include ways in which students are likely to use fractions throughout their lives; it is important for them to understand the connection between these applied uses of fractions and the concepts and procedures involving fractions that they learn in the classroom.

- **Recommendation 5** urges teacher education and professional development programs to emphasize how to improve students' understanding of fractions and to ensure that teachers have sufficient understanding of fractions to achieve this goal. Far too many teachers have difficulty explaining interpretations of fractions other than the part-whole interpretation, which is useful in some contexts but not others. Although many teachers can describe conventional algorithms for solving fractions problems, few can justify them, explain why they yield correct answers, or explain why some nonstandard procedures that students generate yield correct answers despite not looking like a conventional algorithm. Greater understanding of fractions, knowledge of students' conceptions and misconceptions about fractions, and effective practices for teaching fractions are critically important for improving classroom instruction.

Use of research

The recommendations in this practice guide are based on numerous types of evidence, including national and international assessments of students' mathematical knowledge, a survey of teachers' views of the greatest problems in their students' preparation for learning algebra, mathematicians' analyses of key concepts for understanding fractions, descriptive studies of successful and unsuccessful fractions learners, and controlled experimental evaluations of interventions designed to improve learning of fractions.

The research base for the guide was identified through a comprehensive search for studies over the past 20 years that evaluated teaching and learning about fractions. This search was done for a large number of keywords related to fractions teaching and learning that were suggested by the panel members; the results were supplemented by specific studies known to panel members that were not identified by the database search, including earlier works. The process yielded more than 3,000 citations. Of these, 132 met the WWC criteria for review, 33 met the causal validity standards of the WWC, and 20 of the 33 met the causal validity standards of the WWC and were relevant to the panel's recommendations.

In some cases, recommendations are based on such rigorous research. But when research was rare or did not meet WWC standards, the recommendations reflect what this guide's panel believes are best practices, based on instructional approaches having been successfully implemented in case studies or in curricula that have not been rigorously evaluated. The panel could not fulfill its wish to base all recommendations on studies that met WWC standards, in large part because far less research is available on fractions than on development of skills and concepts regarding whole numbers. For example, the *2nd Handbook of Research on Mathematics Teaching and Learning* (National Council of Teachers of Mathematics, 2007) includes 109 citations of research published in 2000 or later on whole numbers but only nine citations of research on fractions published over the same period. High-quality studies testing the effectiveness of specific instructional techniques with fractions were especially scarce. A greater amount of high-quality research on fractions is clearly needed, especially studies that compare the effectiveness of alternative ways of teaching children about fractions.

Table 2 shows each recommendation and the strength of the evidence that supports it as determined by the panel. Following the recommendations and suggestions for carrying out the recommendations, Appendix D presents more information on the research evidence that supports each recommendation.

Table 2. Recommendations and corresponding levels of evidence

	Levels of Evidence		
Recommendation	**Minimal Evidence**	**Moderate Evidence**	**Strong Evidence**
1. Build on students' informal understanding of sharing and proportionality to develop initial fraction concepts.	◆		
2. Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.		◆	
3. Help students understand why procedures for computations with fractions make sense.		◆	
4. Develop students' conceptual understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems.	◆		
5. Professional development programs should place a high priority on improving teachers' understanding of fractions and of how to teach them.	◆		

Recommendation 1

Build on students' informal understanding of sharing and proportionality to develop initial fraction concepts.

Students come to kindergarten with a rudimentary understanding of basic fraction concepts. They can share a set of objects equally among a group of people (i.e., equal sharing)[21] *and identify equivalent proportions of common shapes (i.e., proportional reasoning).*[22]

By using this early knowledge to introduce fractions, teachers allow students to build on what they already know. This facilitates connections between students' intuitive knowledge and formal fraction concepts. The panel recommends using sharing activities to develop students' understanding of ordering and equivalence relations among fractions.

Sharing activities can introduce children to several of the basic interpretations of fractions discussed in the introduction. Sharing can be presented in terms of division—such as by partitioning 12 candies into four equally numerous groups. Sharing also can be presented in terms of ratios; for example, if three cakes are shared by two children, the ratio of the number of cakes to the number of children is 3:2.

Although fractions are typically introduced by 1st or 2nd grade, both the sharing and the proportional reasoning activities described in this recommendation can begin as early as preschool or kindergarten.

Summary of evidence: Minimal Evidence

This recommendation is based on studies showing that students have an early understanding of sharing and proportionality,[23] and on studies of instruction that use sharing scenarios to teach fraction concepts.[24] However, none of the studies that used sharing scenarios to teach fraction concepts met WWC standards. Despite the limited evidence, the

panel believes that students' informal knowledge of sharing and proportionality provides a foundation for introducing and teaching fraction concepts.

Equal sharing. Children have an early understanding of how to create equal shares. By age 4, children can distribute equal numbers of equal-size objects among a small number of recipients, and the ability to equally share improves with age.[25] Sharing a set of discrete objects (e.g., 12 grapes shared among three children) tends to be easier for young children than sharing a single object (e.g., a candy bar), but by age 5 or 6, children are reasonably skilled at both.[26]

Case studies show how an early understanding of sharing could be used to teach fractions to elementary students.[27] In two studies, teachers posed story problems with sharing scenarios to teach fraction concepts such as equivalence and ordering, as well as fraction computation. The studies reported positive effects on fraction knowledge, but they do not provide rigorous evidence on the impact of instruction based on sharing activities.

Proportional relations. The panel believes that instructional practices can build on young children's rudimentary knowledge of proportionality to teach fraction concepts. This early understanding of proportionality has been demonstrated in different ways. By age 6, children can match equivalent proportions represented by different geometric figures and by everyday objects of different shapes.[28] One-half is an important landmark in comparing proportions; children more often succeed on comparisons in which one proportion is more than half and the other is less than half, than on comparisons in which both proportions are more than half or both are less than half (e.g., comparing $\frac{1}{3}$ to $\frac{3}{5}$ is easier than comparing $\frac{2}{3}$ to $\frac{4}{5}$).[29] In addition, children can complete analogies based on proportional relations—for example, half circle is to half rectangle as quarter circle is to quarter rectangle.[30]

Although there is evidence that describes young children's knowledge of proportionality, no rigorous studies that met WWC standards have examined whether this early-developing knowledge can be used to improve teaching of fraction concepts.

How to carry out the recommendation

1. Use equal-sharing activities to introduce the concept of fractions. Use sharing activities that involve dividing sets of objects as well as single whole objects.

The panel recommends that teachers offer a progression of sharing activities that builds on students' existing strategies for dividing objects. Teachers should begin with activities that involve equally sharing a set of objects among a group of recipients and progress to sharing scenarios that require partitioning an object or set of objects into fractional parts. In addition, early activities should build on students' halving strategy (dividing something into two equal sets or parts) before having students partition objects among larger numbers of recipients. Students should be encouraged to use counters (e.g., beans, tokens), create drawings, or rely on other representations to solve these sharing problems; then teachers can introduce formal fraction names (e.g., *one-third*, *one-fourth*, *thirds*, *quarters*) and have children label their drawings to name the shared parts of an object (e.g., $\frac{1}{3}$ or $\frac{1}{8}$ of a pizza). For optimal success, children should engage in a variety of such labeling activities, not just one or two.

Sharing a set of objects. Teachers should initially have students solve problems that involve two or more people sharing a set of objects (see Figure 1). The problems should include sets of objects that can be evenly divided among sharers, so there are no remaining objects that need to be partitioned into fractional pieces.

In these early sharing problems, teachers should describe the number of items and the number of recipients sharing those items, and students should determine how many items each person receives.[31] Teachers might then pose the same problem with increasing numbers of recipients.[32] It is important to emphasize that these problems require sharing a set of objects equally, so that students focus on giving each person the same number of objects.

Partitioning a single object. Next, teachers should pose sharing problems that result in students dividing one or more objects into equal parts. The focus of these problems shifts from asking students *how many* things each person should get to asking students *how much* of an object each person should get. For example, when one cookie is shared between two children, students have to think about how much of the cookie each child should receive.

Teachers can begin with problems that involve multiple people sharing a single object (e.g., four people sharing an apple) and progress to problems with multiple people sharing a set of objects that must be divided into smaller parts to share equally (e.g., three people sharing four apples). Problems that involve sharing one object result in shares that are *unit fractions* (e.g., $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{9}$), whereas scenarios with multiple people and objects often result in *non-unit fractions* (e.g., $\frac{3}{4}$).[33] This distinction between unit and non-unit fractions is important, because when fractions are reduced to lowest terms, non-unit fractions are composed of unit fractions (e.g., $\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$), but the opposite is not the case. Sharing situations that result in unit fractions provide a useful starting point

Figure 1. Sharing a set of objects evenly among recipients

Problem

Three children want to share 12 cookies so that each child receives the same number of cookies. How many cookies should each child get?

Examples of Solution Strategies

Students can solve this problem by drawing three figures to represent the children and then drawing cookies by each figure, giving one cookie to the first child, one to the second, and one to the third, continuing until they have distributed 12 cookies to the three children, and then counting the number of cookies distributed to each child. Other students may solve the problem by simply dealing the cookies into three piles, as if they were dealing cards.

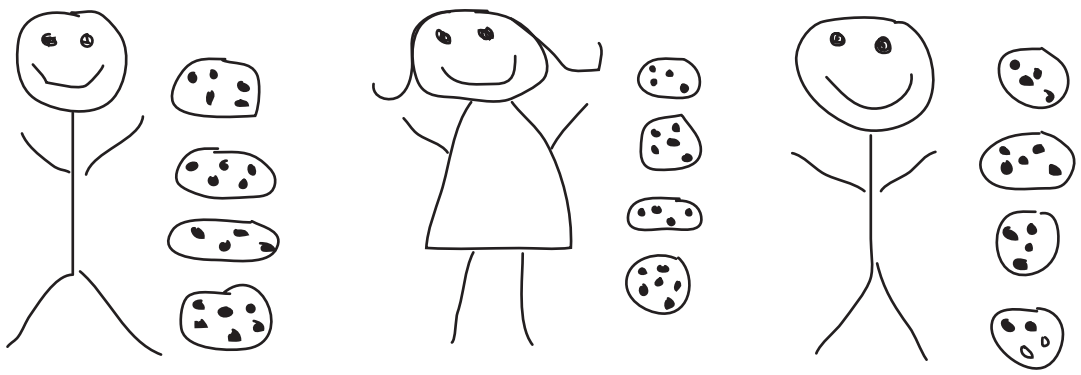


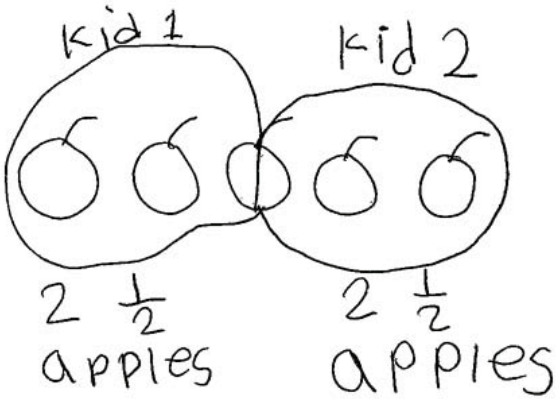
Figure 2. Partitioning both multiple and single objects

Problem

Two children want to share five apples that are the same size so that both have the same amount to eat. Draw a picture to show what each child should receive.

Examples of Solution Strategies

Students might solve this problem by drawing five circles to represent the five apples and two figures to represent the two children. Students then might draw lines connecting each child to two apples. Finally, they might draw a line partitioning the final apple into two approximately equal parts and draw a line from each part to the two children. Alternatively, as in the picture to the right, children might draw a large circle representing each child, two apples within each circle, and a fifth apple straddling the circles representing the two children. In yet another possibility, children might divide each apple into two parts and then connect five half apples to the representation of each figure.



for introducing fraction names, especially because some children think that all fractional parts are called one-half.[34]

The panel also suggests starting with problems that involve sharing among two, four, or eight people (i.e., powers of two).[35] This allows students to create equal parts by using a halving strategy—dividing an object in half, dividing the resulting halves in half, and so on, until there are enough pieces to share (see Figure 2).[36] Eventually, students should solve sharing problems for which they cannot use a halving strategy. Partitioning a brownie into thirds, for example, requires that students anticipate how to slice the brownie so that it results in three equal parts. Students may be tempted to use repeated halving for all sharing problems, but teachers should help students develop other strategies for partitioning an object. One approach is to have students place wooden sticks on concrete shapes, with the sticks representing the slices or cuts that a student would make to partition the object.[37]

2. Extend equal-sharing activities to develop students' understanding of ordering and equivalence of fractions.

Teachers can extend the types of sharing activities described in the previous step to develop students' understanding of ordering and identifying equivalent fractions. The overall approach remains the same: teachers pose story problems that involve a group of people sharing objects, and students create drawings or other representations to solve the problems. However, teachers use scenarios that require fraction *comparisons* or identification of equivalent fractions and focus on different aspects of students' solutions.

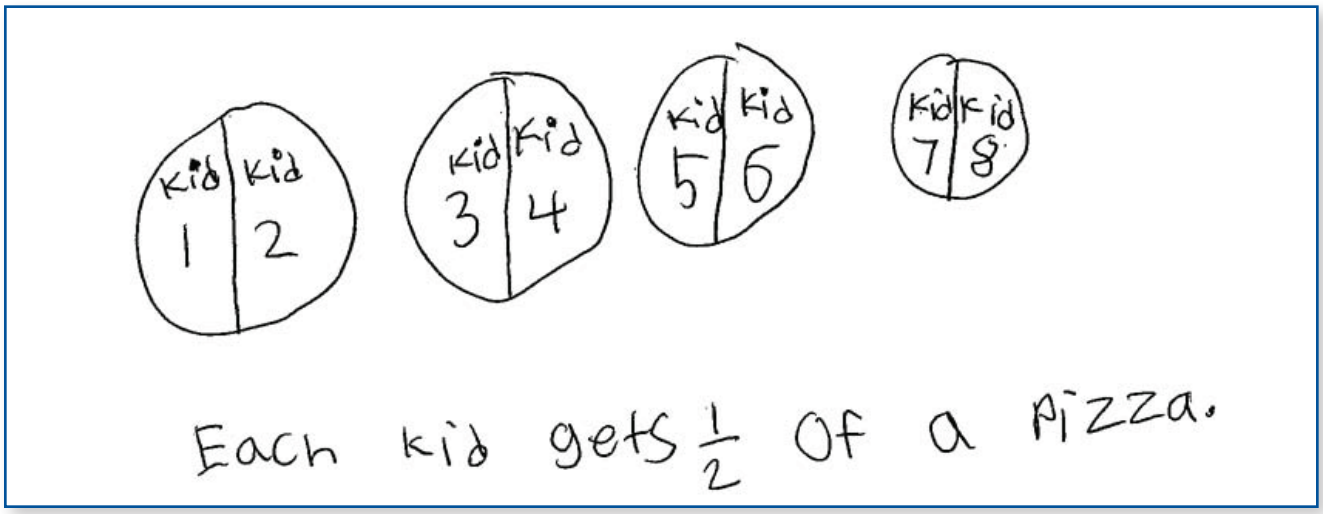
Sharing activities can be used to help students understand the relative size of fractions. Teachers can present sharing scenarios with an increasing number of recipients and have students compare the relative size of each resulting share. For example, students can compare the size of pieces that result when sharing a candy bar equally among three, four, five, or six children.[38] Teachers should encourage students to notice that as the number of people sharing the objects increases, the size of each person's share decreases; they should then link this idea to formal fraction names and encourage students to compare the fractional pieces using fraction names (e.g., ⅓ of an object is greater than ¼ of it).

When using sharing scenarios to discuss equivalent fractions, teachers should consider two approaches, both of which should be used with scenarios in which the number of sharers and the number of pieces to be shared have one or more common factors (e.g., four pizzas shared among eight children):

- **Partition objects into larger or smaller pieces.** One way to understand equivalent shares is to discuss alternative ways to partition and receive the same shares.[39] Students can think about how to solve a sharing scenario using different partitions to produce equal shares. Such partitioning may require trial and error on the part of students to identify which groupings result in equal shares. Students might combine smaller pieces to make bigger ones or partition bigger ones into smaller pieces. For example, to solve the problem of eight children sharing four pizzas, students might partition all four pizzas into eighths and then give each child four pieces of size ⅛. Alternatively, students could divide each pizza into fourths and give each person ⅔, or divide each pizza into halves and distribute ½ to each child. Students should understand that although there are different ways to partition the pizza, each partitioning method results in equivalent shares.

- **Partition the number of sharers and the number of items.** Another way to help students understand equivalence is to partition the number of sharers and objects.[40] For example, if students arrive at ⅝ for the problem in the previous paragraph, the teacher could ask how the problem would change if the group split into two tables and at each table four children shared two pizzas. Students can compare the new solution of ⅔ to their original solution of ⅝ to show that the two amounts are equivalent (see Figure 3). To drive home the point, the eight children could then sit at four tables, with two children at each table sharing a single pizza—and reaching the more familiar concept of ½.

Figure 3. Student work for sharing four pizzas among eight children



Another way to teach equivalent fractions with sharing scenarios is to pose a missing-value problem in which children determine the number of objects needed to create an equivalent share. For example, if six children share eight oranges at one table, how many oranges are needed at a table of three children to ensure each child receives the same amount?[41] The problem could be extended to tables with 12 children, 24 children, or 9 children. To solve these problems, students might identify how much one child receives in the first scenario and apply that to the second scenario. Alternatively, they could use the strategy described above and partition the six children and eight oranges at the original table into two tables, so that the number of children and oranges at the first new table equal the number of children and oranges at the second new table.

Here is another example that allows students to explore the concept of equal partitioning: if 24 children are going out for sandwiches, and 16 sandwiches have been ordered, what are the different ways the children could sit at tables and divide the sandwiches so they would all receive the same amount? Options might include having one big table of 24 children and 16 sandwiches, having four tables of six children and four sandwiches at each, eight tables of three children and two sandwiches at each, and so on.

3. Build on students' informal understanding to develop more advanced understanding of proportional-reasoning concepts. Begin with activities that involve similar proportions, and progress to activities that involve ordering different proportions.

Early instruction can build on students' informal understanding to develop basic concepts related to proportional reasoning. Teachers should initially pose problems that encourage students to think about the proportional relations between pairs of objects, without necessarily specifying exact quantities. For example, teachers could use the story of *Goldilocks and the Three Bears* to discuss how the big bear needs a big chair, the medium-sized bear needs a medium-sized chair, and the small bear needs a small chair.[42]

The following list provides examples of different relations relevant to early proportional reasoning that can be explored with students:

- **Proportional relations.** Teachers can discuss stories or scenarios that present basic proportional relations that are not quantified. For example, a class could discuss the number of students it would take to balance a seesaw with one, two, or three adults on one end. Creating more and less saturated liquid mixtures with lemonade mix or food coloring can facilitate discussions comparing the strength or concentration of different mixtures.
- **Covariation.** Teachers should discuss problems that involve one quantity increasing as another quantity increases. Examples could include the relation between height and clothing size or between foot length and shoe size.[43]
- **Patterns.** Simple repeating patterns can be useful for discussing the concept of ratio. For example, students could complete a pattern such as blue star, blue star, red square, blue star, blue star, red square, blue star, blue star, red square, and so on.[44] Teachers can then discuss how many blue stars there are for every red square, have students arrange the stars and squares to show what gets repeated, have students change the pattern to a different ratio (e.g., three blue stars to one red square), or have students extend the pattern.[45]

Potential roadblocks and solutions

Roadblock 1.1. *Students are unable to draw equal-size parts.*

Suggested Approach. Let students know that it is acceptable to draw parts that are not exactly equal, as long as they remember that the parts should be considered equal.

Roadblock 1.2. *Students do not share all of the items (non-exhaustive sharing) or do not create equal shares.*

Suggested Approach. Although children have an intuitive understanding of sharing situations, they sometimes make mistakes in their attempts to solve sharing problems. Students may not share all of the items, especially if a sharing scenario requires partitioning an object. Teachers should help students understand that sharing scenarios require sharing all of the objects—possibly even noting that each child wants to receive as much as he or she possibly can, so no objects should remain unaccounted for.

Students also might not create equal shares because they do not understand that dealing out equal-size objects results in an equal amount for each person.[46] In this case, teachers can discuss how dealing out objects ensures that each person receives an equal amount and can encourage students to verify that they divided the items equally.

Equal sharing is important because it lays a foundation for later understanding of equivalent fractions and equivalent magnitude differences (e.g., understanding that the difference between 0 and ½ is the same as the difference between 1 and 1½ or between 73 and 73½).

Roadblock 1.3. *When creating equal shares, students do not distinguish between the number of things shared and the quantity shared.*

Suggested Approach. Younger students in particular may confuse equal numbers of shares with equal amounts shared.[47] For example, if students are asked to provide equal amounts of food from a plate with both big and small pieces, a child might give out equal numbers of pieces of food rather than equal amounts. This misunderstanding may stem from limited experience with situations in which entities of different sizes are dealt out or shared.

One way to address this misconception is to use color cues to help students distinguish between the quantity being shared and the number of items being shared.[48] For example, in a scenario in which both of two identical toy dogs are said to be hungry, children could be asked whether the dogs would have the same amount to eat if one dog received five large red pieces of pretend food and the other dog five small green pieces of pretend food.

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

Early fractions instruction generally focuses on the idea that fractions represent parts of a whole (e.g., one-third as the relation of one part to a whole that has three equal parts). Although the part-whole interpretation of fractions is important, too often instruction does not convey another simple but critical idea: fractions are numbers with magnitudes (values) that can be either ordered or considered equivalent.

Many common misconceptions—such as that two fractions should be added by adding the numerators and then adding the denominators—stem from not understanding that fractions are numbers with magnitudes. Not understanding this can even lead to confusion regarding whether fractions are numbers. For example, many students believe that four-thirds is not a number, advancing explanations such as, "You cannot have four parts of an object that is divided into three parts."[49] Further, many students do not understand that fractions provide a unit of measure that allows more precise measurement than whole numbers; these students fail to realize that an infinite range of numbers exists between successive whole numbers or between any two fractions.[50] Reliance on part-whole instruction alone also leaves unclear how fractions are related to whole numbers.

An effective way to develop students' understanding of fractions as numbers with magnitudes is to use number lines. Number lines can clearly illustrate the magnitude of fractions; the relation between whole numbers and fractions; and the relations among fractions, decimals, and percents. They also provide a starting point for building students' number sense with fractions and provide a way to represent negative fractions visually, which can otherwise be a challenging task. All of these types of understanding are crucial for learning algebra and other more advanced areas of mathematics.

Summary of evidence: Moderate Evidence

Evidence for this recommendation primarily comes from studies demonstrating the usefulness of number lines for developing number sense with whole numbers. These studies used number line representations to teach preschool and early elementary students about the magnitudes of whole numbers.[51] An additional study showed how number lines can be used to teach decimals successfully.[52] All of these studies met WWC evidence standards. Moreover, accuracy in locating whole numbers on number lines is related to mathematical achievement among students in kindergarten through 4th grade, and accuracy in locating decimals on number lines is related to classroom mathematics grades among 5th- and 6th-graders.[53] The panel believes that given the applicability of number lines to fractions as well as whole numbers, these findings indicate that number lines can improve learning of fractions in elementary and middle school.

Number lines with whole numbers. Playing a linear board game with whole numbers for about one hour (four 15-minute sessions over a two-week period) improved understanding of numerical magnitudes by preschoolers from low-income backgrounds.[54] The game involved moving a marker one or two spaces at a time across a horizontal board that had the numbers 1 to 10 listed in order from left to right in consecutive squares. Two additional studies showed the value of other number line procedures for improving knowledge of whole number magnitudes. Estimating the locations of 10 numbers on a 0-to-100 number line improved 1st-graders' ability to locate whole numbers on the number line;[55] and showing 1st-grade students the addends and sums of addition problems on a number line increased the likelihood that students correctly answered the problems later.

Number lines with decimals. In another study, number lines were used to teach decimal concepts to 5th- and 6th-grade students.[56] The teaching technique involved providing students with practice locating decimals on a number line divided into tenths and with a prompt to notice the tenths digit for each number. These students were later more accurate in locating decimals on a number line than students whose number lines were not divided into tenths and did not receive prompts. For all students in the study, a before-and-after comparison showed that conceptual understanding of fractions improved after locating decimals on a number line. This last finding is suggestive evidence, because there is no comparison group of students who did not use a number line.

Another study examined a Dutch curriculum that used number lines and measurement contexts to teach fractions.[57] Students in the treatment group located and compared fractions on a number line and measured objects in the classroom using a strip that could be folded to measure fractional parts. Although this study did not meet WWC evidence standards, the authors reported positive effects on middle school students' number sense with fractions.[58] Two additional studies that were not eligible for review found mixed results of using a number line to teach fraction concepts. Both studies noted challenges that students face in understanding fractions on number lines.[59] For example, one study reported that students had difficulty finding equivalent fractions on

a number line partitioned into smaller units (e.g., finding ⅓ on a number line divided into sixths).[60]

Other evidence that is consistent with the recommendation includes a study showing the relation between skill at estimating locations of decimals on a number line and math grades for 5th- and 6th-grade students,[61] and a mathematician's analysis indicating that learning to represent the full range of numbers on number lines is fundamental to understanding numbers.[62]

How to carry out the recommendation

1. Use measurement activities and number lines to help students understand that fractions are numbers, with all the properties that numbers share.

When students view fractions as numbers, they understand that fractions, like whole numbers, can be used to measure quantities. Measurement activities provide a natural context in this regard.[63] Through such activities, teachers can develop the idea that fractions allow for more precise measurement of quantities than do whole numbers.

Teachers can present situations in which fractions are used to solve problems that cannot be solved with whole numbers. For example, they can ask students how to describe the amount of sugar in a cookie recipe that needs more than 1 cup but less than 2 cups.

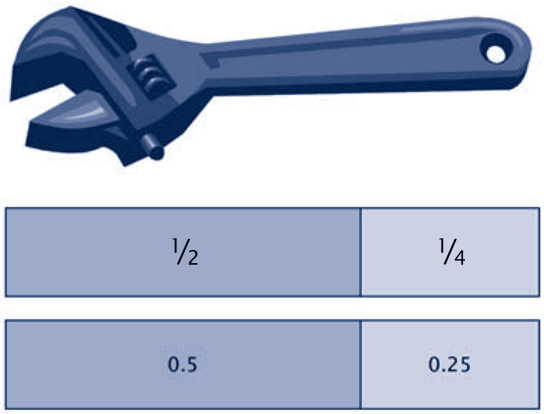
Teachers can then show students the various measurement lines on a measuring cup and convey the importance of fractions in describing quantities. Teachers should emphasize that fractions provide a more precise unit of measure than whole numbers and allow students to describe quantities that whole numbers cannot represent. Fraction strips (also known as fraction strip drawings, strip diagrams, bar strip diagrams, and tape diagrams) are length models that allow students to measure objects using fractional parts and reinforce the idea that fractions can be used to represent quantities (see Example 1).

Example 1. Measurement activities with fraction strips

Teachers can use fraction strips as the basis for measurement activities to reinforce the concept that fractions are numbers that represent quantities.[64]

To start, students can take a strip of card stock or construction paper that represents the initial unit of measure (i.e., a whole) and use that strip to measure objects in the classroom (desk, chalkboard, book, etc.). When the length of an object is not equal to a whole number of strips, teachers can provide students with strips that represent fractional amounts of the original strip. For example, a student might use three whole strips and a half strip to measure a desk.

Using fraction strips to measure an object



Teachers should emphasize that fraction strips represent different units of measure and should have students measure the same object first using only whole strips and then using a fractional strip. Teachers should discuss how the length of the object remains the same but how different units of measure allow for better precision in describing it. Students should realize that the size of the subsequently presented fraction strips is defined by the size of the original strip (i.e., a half strip is equal to one-half the length of the original strip).

2. Provide opportunities for students to locate and compare fractions on number lines.

Teachers should provide opportunities for students to locate and compare fractions on number lines. These activities should include fractions in a variety of forms, including proper fractions (2/3), improper fractions (5/3), mixed numbers (1 2/3), whole numbers (4/2), decimals (0.40), and percents (70%).

Teachers can initially have students locate and compare fractions on number lines with the fractions already marked (e.g., a number line with marks indicating tenths). Pre-segmented number lines avoid the difficulty students have in accurately partitioning the number line. These number lines also are useful for locating and comparing fractions whose locations are indicated (e.g., 3/8 and 5/8 on a number line with eighths marked) and fractions whose denominator is a factor of the unit fractions shown on the number line (e.g., 1/4 and 3/4 on a line with eighths marked), as well as fractions with other denominators (e.g., 1/7, 3/5). For example, students might compare the locations of 7/8 and 3/4 on a number line marked with eighths. These activities should include opportunities for students to locate whole numbers on the number line and compare their locations to those of fractions, including ones equivalent to whole numbers (e.g., locating 1 and 8/8).

Number lines also can be used to compare fractions of varying sizes to whole numbers greater than one (locating 10/3 on a number line with 0 at the left end, 5 at the right end, and 1, 2, 3, and 4 marked in between). Example 2 provides a strategy that can be used to introduce students to the idea of locating fractions on a number line.

Comparing fractions with different denominators on a pre-segmented number line can be complicated for young students—for example, comparing 3/8 and 1/3 on a number line divided into eighths. To help students understand such problems, teachers can label number lines with one fractional-unit sequence above the number line and a different fractional-unit sequence below the number line. For example, when asking students to compare 1/3 and 3/8, teachers might label eighths above the number line and thirds below it. Such number lines allow students who are relatively early in the process of learning about fractions to locate and compare fractions with different denominators and to think about the relative size of the fractions.[66]

Example 2. Introducing fractions on a number line

The following example describes one way to introduce the idea of locating fractions on a number line, emphasizing that fractions are numbers with quantities.

To illustrate the location of 3/5 on a 0-to-5 number line, the teacher might first mark and label the location of 1 and then divide the space between each whole number into five equal-size parts. After this, the teacher might add the labels 0/5, 1/5, 2/5, 3/5, 4/5, and 5/5 in the 0–1 part of the number line and highlight the location of 3/5.[65] Displaying whole numbers as fractions (e.g., 5/5) allows teachers to discuss what it means to describe whole numbers in terms of fractions and to clarify that whole numbers are fractions too.

Teachers also should provide students with opportunities to locate and compare fractions on number lines that are minimally labeled—for example, ones with the labels 0, 1/2, 1, 1 1/2, and 2. This approach is almost a necessity for fractions with large denominators (e.g., dividing a number line into 28ths is difficult) and encourages students to think about the location of fractions relative to the labeled landmarks.[67] For example, teachers can have students locate 6/7 on a number line marked with 0, 1/2, and 1.

For a whole-class activity, teachers can draw a number line on the board and have students mark estimates of where different fractions

fall. As the number line fills up, teachers can guide a discussion about fractions yet to be placed, highlighting the need to preserve the correct order. Inserting decimals and percentages on that same number line can teach additional valuable lessons.

Finally, teachers should encourage students to think about the distance between two fractions: For example, students could compare 1/12 and 1/4 and consider whether 1/12 is closer to 1/4 or 0. Similarly, 0.3 or 0.45 could be compared to locations marked 0, 1/2, and 1, or 0, 0.5, and 1.

3. Use number lines to improve students' understanding of fraction equivalence, fraction density (the concept that there are an infinite number of fractions between any two fractions), and negative fractions.

In addition to being useful for comparing positive fraction magnitudes, number lines also can be valuable for teaching equivalent fractions, negative fractions, and fraction density. Number lines are, of course, not the only way to teach these concepts, but the panel believes they are helpful for improving students' understanding.

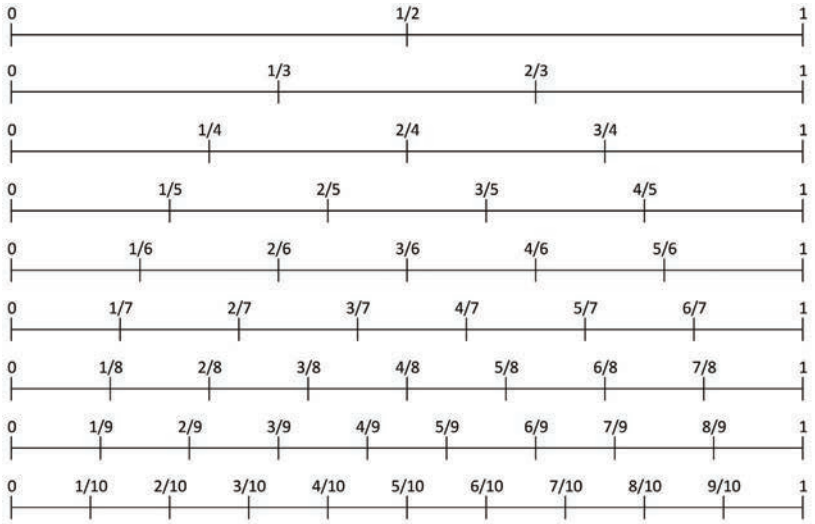
Number lines can be used to illustrate that equivalent fractions describe the same magnitude. For example, asking students to locate 2/5 and 4/10 on a single number line can help them understand the equivalence of these numbers. Teachers can mark fifths above the line and tenths below it (or vice versa) to help students with this task. Although viewing equivalent fractions as the same point on a number line can be challenging for students,[68] the panel believes that the ability to do so is critical for thorough understanding of fractions.

A discussion of equivalent fractions should build on points made in Step 1 about fractions on the number line. For example, teachers can divide a 0-to-1 number line into halves and quarters and show that 1/2 and 2/4 occupy the same, or equivalent, point on the number line (see Figure 4). Students can use a ruler to identify equivalent fractions on the stacked number lines shown in Figure 4, identifying fractions that occupy the same location on each number line. Fraction strips also can be used to reinforce the concept of equivalent fractions by allowing students to measure the distance between two points using different-sized fraction strips (see Figure 5).

Number lines also can be used to help students understand that an infinite number of fractions exist between any two other fractions. This is one way in which fractions differ

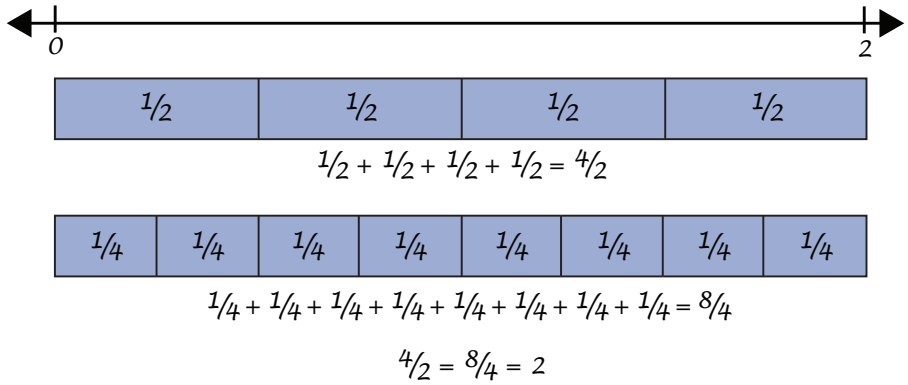
Figure 4. Finding equivalent fractions on a number line

Use of number lines to teach equivalence of fractions in a Japanese curriculum



Source: Adapted from Shoseki (2010).

Figure 5. Using fraction strips to demonstrate equivalent fractions



from whole numbers and can be a difficult concept for students to grasp.[69] Teachers can help students understand this concept by asking them to make successive partitions on the number line, creating smaller and smaller unit fractions.[70] For example, students can divide whole number segments in half to create halves, and then divide each half into halves to create fourths, then divide each fourth into halves to create eighths, and so on (this activity also can be done with thirds, ninths, twenty-sevenths, etc.). Such divisions show students that they always can partition a number line using smaller unit fractions.[71] The same can be done with decimals and percents—such as by showing that 0.13, 0.15, and 0.17 are among the infinite numbers that fall between 0.1 and 0.2, and that 2% falls between 0% and 10%.

The panel further recommends that teachers use number lines when introducing negative fractions. Teaching negative fractions in a part-whole context can be difficult, because the idea of a negative part of a whole is non-intuitive. But the number line provides a straightforward visual representation of fractions less than zero, as well as fractions greater than zero.

By providing number lines that include marks and labels for zero, for several positive fractions, and for several negative fractions with the same absolute values as the positive fractions, teachers can help convey the symmetry about zero of positive and negative fractions. And by placing positive and negative fractions into stories—possibly about locations above and below sea level or about money gained or lost—teachers can illustrate addition and subtraction of both types of fractions.

4. Help students understand that fractions can be represented as common fractions, decimals, and percentages, and develop students' ability to translate among these forms.

Students need a broad view of fractions as numbers. That includes understanding that fractions can be represented as decimals and percents as well as common fractions. Teachers should clearly convey that common fractions, decimals, and percents are just different ways of representing the same number.

Number lines provide a useful tool for helping students understand that fractions, decimals, and percents are different ways of describing the same number. By using a number line with common fractions listed above it and decimals or percentages below it, teachers can help students locate and compare fractions, decimals, and percents on the same number line. For example, teachers can provide students with a number line marked with 0 and 1, and students can be asked to locate ¾, 0.75, and 75% on it. In addition, when students use division to translate a fraction into a decimal, they can plot both the fraction and the decimal on the same number line.

Potential roadblocks and solutions

Roadblock 2.1. *Students try to partition the number line into fourths by drawing four hash marks rather than three, or they treat the whole number line as the unit.*[72]

Suggested Approach. When using a number line with fractions, students must be taught to represent fourths as four equal-size segments between two whole numbers. Teachers should demonstrate that inserting three equally spaced hash marks between, say, 0 and 1 divides the space into four equal segments, or fourths. This rule can be generalized so that students know that dividing the number line into $1/n$ units requires drawing $n - 1$ hash marks between two whole numbers.

Roadblock 2.2. *When students locate fractions on the number line, they treat the numbers in the fraction as whole numbers (e.g., placing ¾ between 3 and 4).*

Suggested Approach. This mistake reflects a common misconception in which students apply their whole number knowledge to fractions—viewing the numbers that make up a fraction as separate whole numbers. The misconception can be addressed by presenting students with contrasting cases: for example, having them locate 3 and 4 on a 0-to-4 number line, then identifying ¾ as a fraction between 0 and 1, and finally discussing why each fraction goes where it is placed.

Roadblock 2.3. *Students have difficulty understanding that two equivalent fractions are the same point on a number line.*

Suggested Approach. Students often have trouble internalizing how partitions that locate one fraction (e.g., eighths partitions for locating ⅝) also can help locate an equivalent fraction (e.g., ½). One way to address this lack of understanding is to show students one set of numerical labels above the number line and another set of labels below it. Thus, halves could be marked just above the line and eighths just below it, and teachers could point out the equivalent positions of ½ and ⅝, of 1 and ⅜, of 1½ and 1⅝, and so on. Another approach is for students to create a number line showing ½ and another number line showing ⅝ and then compare the two. Teachers can line up the two number lines and lead a discussion about equivalent fractions.

Roadblock 2.4. *The curriculum materials used by my school district focus on part-whole representations and do not use the number line as a key representational tool for fraction concepts and operations.*

Suggested Approach. Although it is important for students to understand that fractions represent parts of a whole, the panel notes that this is only one use of fractions and therefore recommends the use of number lines and measurement contexts to develop a comprehensive understanding of fractions. Manipulatives that often are used to represent part-whole interpretations, such as fraction circles and fraction strips, also can be used to convey measurement interpretations, but considerable care needs to be taken to avoid students simply counting parts of the fraction strip or circle that correspond to the numerator and to the denominator without understanding how the numerator and denominator together indicate a single quantity. Using number lines that are unmarked between the endpoints can avoid such counting without understanding. Some textbooks use number lines extensively for teaching fractions; teachers should examine those books for ideas about how to use number lines to convey the idea that fractions are measures of quantity.

Recommendation 3

Help students understand why procedures for computations with fractions make sense.

Students are most proficient at applying computational procedures when they understand why those procedures make sense. Although conceptual understanding is foundational for the correct use of procedures, students often are taught computational procedures with fractions without an adequate explanation of how or why the procedures work.

Teachers should take the time to provide such explanations and to emphasize how fraction computation procedures transform the fractions in meaningful ways. In other words, they should focus on both conceptual understanding and procedural fluency and should emphasize the connections between them. The panel recommends several practices for developing understanding of computational procedures, including use of visual representations and estimation to reinforce conceptual understanding. Addressing students' misconceptions and setting problems in real-world contexts also can contribute to improved understanding.

Summary of evidence: Moderate Evidence

The panel based this recommendation in large part on three well-designed studies that demonstrated the effectiveness of teaching conceptual understanding when developing students' computational skill with fractions.[73] These studies focused on decimals and were relatively small in scale; however, the panel believes that their results, together with extensive evidence showing that meaningful information is remembered much better than meaningless information, provide persuasive evidence for this recommendation.[74] Additional support for the recommendation comes from four studies that showed a positive relation between conceptual and computational knowledge of fractions.[75]

The studies that contributed to the evidence base for this recommendation used computer-based interventions to examine the link

between conceptual knowledge and computational skill with decimals. Sixth-grade students completed three lessons on decimal place value (i.e., conceptual knowledge) and three lessons on addition and subtraction of decimals (i.e., procedural knowledge).[76] Iterating between the two types of lessons improved students' procedural knowledge, compared with teaching all of the conceptual lessons before any of the procedural ones. In another study, 5th- and 6th-grade students practiced locating decimals on a number line using a computer-based game. Dividing the number line into tenths and encouraging students to notice the tenths digit improved 5th- and 6th-grade students' ability to locate decimals on a number line (compared to not providing the prompts).[77]

Research also shows a positive relationship between students' conceptual and procedural knowledge of fractions. That is, children who have above-average conceptual knowledge also tend to have above-average knowledge of computational procedures. Studies of 4th- and 5th-graders and of 7th- and 8th-graders indicated that conceptual knowledge was positively related to computational proficiency after controlling for prior math achievement, arithmetic fluency, working memory, and reading ability.[78] In addition, conceptual knowledge of decimals predicted students' ability to locate decimals on a number line.[79] While these studies show a correlation between conceptual and procedural knowledge, they did not examine the effectiveness of interventions that develop conceptual knowledge to improve procedural knowledge.

The panel also identified evidence that specifically addressed two of the four steps for implementing this recommendation.

Use of representations. Evidence identified by the panel supports the recommended practice of using visual representations and manipulatives during instruction on fraction computation (Step 1). Two well-designed studies found that the use of manipulatives and pictorial representations had a positive effect on computational skill with fractions.[80] One of these studies focused on fraction circles (sets of circles, in which the first is a whole circle, the second is divided in half, the third is divided in thirds, etc.).[81] The other study had students use a variety of manipulatives for learning computational procedures with fractions, including fraction squares and fraction strips.[82] A third study examined the Rational Number Project curriculum, which emphasizes the use of manipulatives as one of many components.[83] The authors of the study reported that the curriculum had a positive effect on fraction computation abilities. However, manipulatives were only one component of this multifaceted curriculum, and the study provided insufficient information for the WWC to complete a review, so the conclusions that can be drawn from the study regarding the role of manipulatives are limited.

Real-world contexts. The panel identified evidence related to the use of real-world contexts for improving skill at executing computational procedures with fractions (Step 4).[84] In one of the studies, personalizing problems for 5th- and 6th-grade students improved their ability to solve division problems with fractions.[85] The other study found that posing problems in everyday contexts improved 11- and 12-year-old students' ability to order and compare decimals.[86] Additional studies argued for the use of real-world contexts for teaching procedures for computing with fractions but did not provide rigorous evidence that such instruction causes improvement in fraction computation.[87]

How to carry out the recommendation

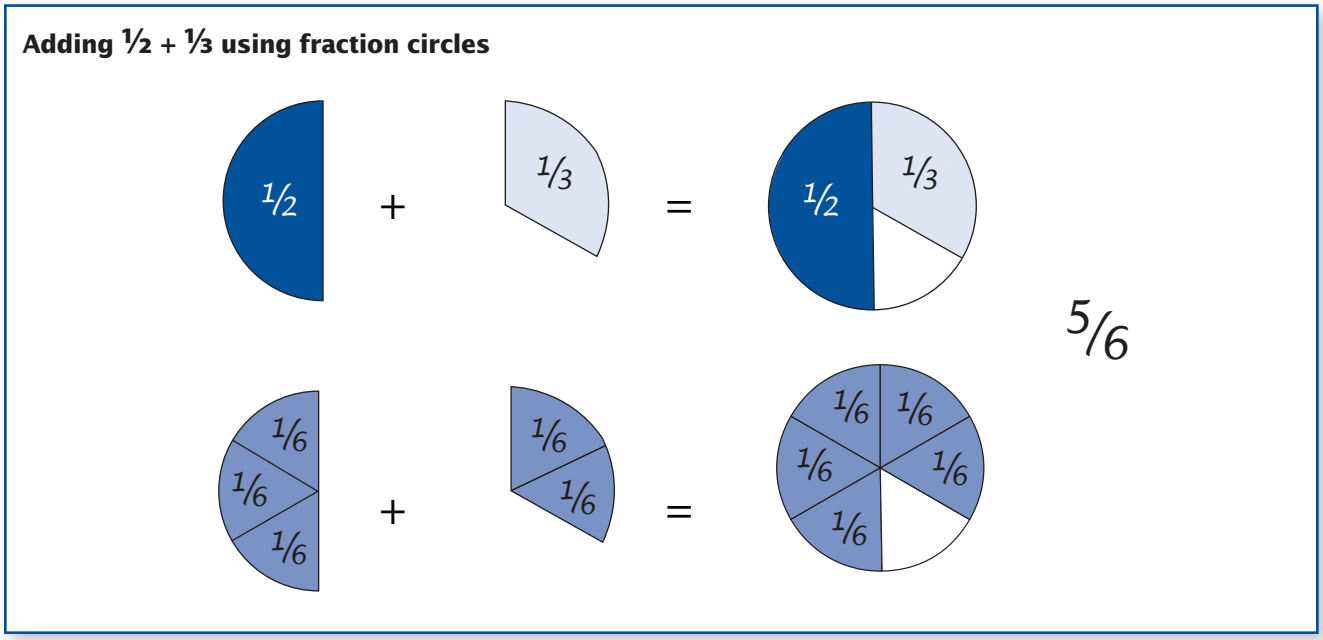
1. Use area models, number lines, and other visual representations to improve students' understanding of formal computational procedures.

Teachers should use visual representations and manipulatives, including number lines and area models, that help students gain insight into basic concepts underlying computational procedures and the reasons why these procedures work. For example, when teaching addition or subtraction of fractions with unlike denominators, teachers should use a representation that helps students see the need for common denominators.

There are several ways teachers can use representations to illuminate key underlying concepts:

- **Find a common denominator when adding and subtracting fractions.** A common mistake students make when faced with fractions that have unlike denominators is to add both numerators and denominators.[88] Certain representations can provide visual cues to help students see the need for common denominators. For example, teachers can demonstrate that when adding pieces corresponding to fractions of objects (e.g., adding ½ of a circle and ⅓ of a circle), converting both ½ and ⅓ to sixths provides a common denominator that applies to both fractions and allows them to be added (Figure 6). Discuss with students why multiplying denominators always indicates a common denominator that can be used to express both original fractions.
- **Redefine the unit when multiplying fractions.** Multiplying two fractions requires finding a fraction of a fraction. For example, when multiplying ¼ by ⅔, students could start with ⅔ of the original (usually unmentioned) unit and find ¼ of this fractional amount. Pictorial or concrete representations can help students visualize this process to improve their understanding of the multiplication procedure. For example, students can shade in with

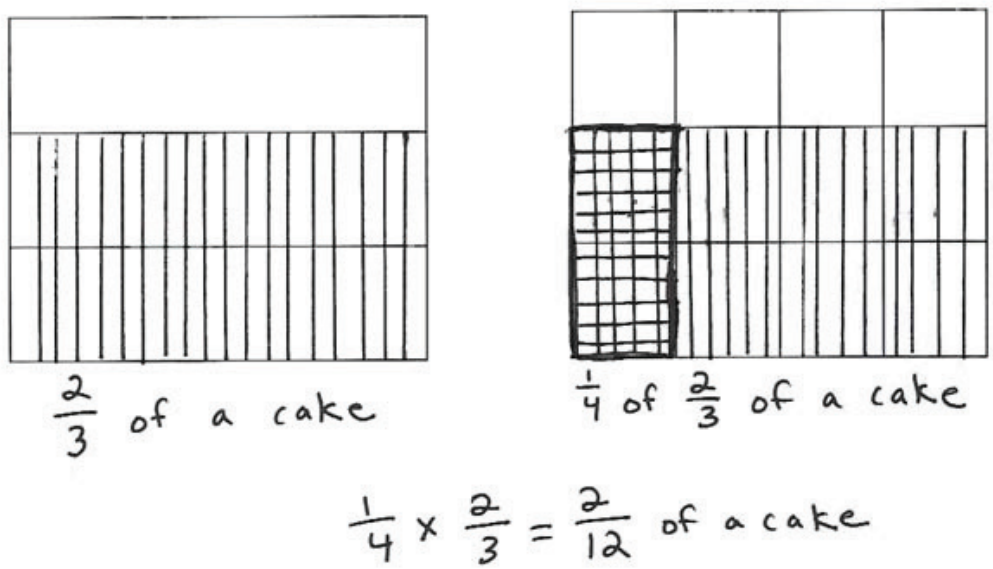
Figure 6. Fraction circles for addition and subtraction



Source: Adapted from Cramer and Wyberg (2009).

Figure 7. Redefining the unit when multiplying fractions

Lori is icing a cake. She knows that 1 cup of icing will cover ⅔ of a cake. How much cake can she cover with ¼ cup of icing?



vertical lines ⅔ of a square cake drawn on paper and then shade in with horizontal lines ¼ of the cake's shaded area, resulting in a product represented by the cross-hatched area (Figure 7).[89] This approach illustrates how to redefine the unit—initially treating the full cake as the whole, and then treating the vertically shaded portion of the cake as the whole.

- **Divide a number into fractional parts.** Dividing fractions is conceptually similar to dividing whole numbers, in that students can think about how many times the divisor goes into the dividend. For example, ½ ÷ ¼ can be represented in terms of "How many ¼s are in ½?"

  Teachers can use representations such as ribbons or a number line to help students model the division process for fractions. Students using ribbons can cut two ribbons of equal size and then separate one into fourths and one into halves. To show the division problem ½ ÷ ¼, students can find out how many fourths of a ribbon fit onto one-half of a ribbon, when the whole ribbon was the same length in both cases (see Figure 8).[90] Similarly, a teacher can draw a number line with both fourths and halves labeled to show students that there are two ¼ segments in ½. Teachers can help students deepen their understanding of the division process by presenting problems in which the divisor, dividend, or both are greater than one, and problems in which the quotient is not an integer, such as 1¾ divided by ½.

Teachers should consider the advantages and disadvantages of different representations for teaching procedures for computing with fractions. A key issue is whether the representation adequately reflects the computation process being taught, allowing students to make links between the two.

Teachers also should think about whether a representation can be used with different types of fractions—proper fractions (⅝),

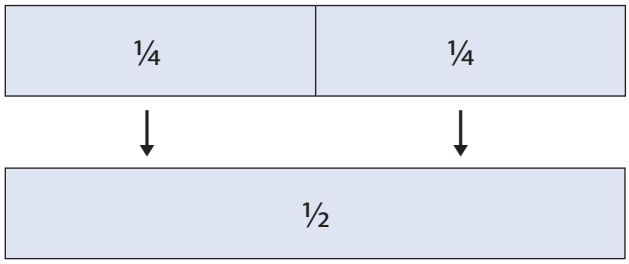
Figure 8. Using ribbons to model division with fractions

Students use ribbons to solve $\frac{1}{2} \div \frac{1}{4}$

Step 1. Divide a ribbon into fourths.

Step 2. Divide a ribbon of the same length into halves.

Step 3. Find out how many fourths of a ribbon can fit into one-half of the ribbon.



Two fourths fit into one-half of the ribbon.
So, $\frac{1}{2} \div \frac{1}{4} = 2$.

mixed numbers ($1\frac{3}{8}$), improper fractions ($\frac{11}{8}$), and negative numbers ($-\frac{1}{2}$). For example, area models may readily illustrate addition of fractions with positive numbers but do not as easily lend themselves to explaining addition of fractions with negative numbers. In contrast, number lines can be used to explain both.

Representations that students have used to learn other mathematical concepts, especially other fraction concepts, may be particularly useful. For example, many students learn to represent decimals using base-10 blocks or 100 grids (10 by 10 squares, with each square representing $\frac{1}{100}$ and the whole square representing 1). Familiarity with this representation also might help students understand adding and subtracting decimal and common fractions. For example, 100 grids can be used to illustrate that adding 2.34 + 1.69 is the same as adding $2\frac{34}{100} + 1\frac{69}{100}$.

2. Provide opportunities for students to use estimation to predict or judge the reasonableness of answers to problems involving computation with fractions.

When teaching procedures for computing with fractions, teachers should provide opportunities for students to estimate the solutions to problems. Estimation requires students to use reasoning skills and thus leads them to focus on the meaning of procedures for computing with fractions.[91] Teachers can ask students to provide an initial estimate and to explain their thinking before having them compute the answer.[92] Students, in turn, can use the estimates to judge the reasonableness of their answers.

To improve estimation skills, teachers can discuss whether and why students' solutions to specific problems are reasonable; they also can ask students to explain the strategies they used to arrive at their estimates and

compare their initial estimates to the solutions they reached by applying a computational algorithm. Consider an example: a student might estimate that the solution of $\frac{1}{2} + \frac{1}{5}$ is more than $\frac{1}{2}$ but less than $\frac{3}{4}$, since $\frac{1}{5}$ is smaller than $\frac{1}{4}$. If the student then incorrectly adds the numerators and denominators to produce the sum $\frac{2}{7}$, the teacher can note that this answer cannot be right because $\frac{2}{7}$ is less than $\frac{1}{2}$.[93] From there, the teacher can guide the student to identify, understand, and correct the procedural error.

Estimation is likely to be most useful with problems in which a solution cannot be computed quickly or easily. There is no point asking students to estimate the answer to a problem that can be solved quickly and accurately by mental computation, such as $\frac{7}{9} - \frac{5}{9}$.

Teaching students effective estimation strategies (Example 3) can maximize the value of estimation for deepening understanding of computations involving fractions.

Example 3. Strategies for estimating with fractions

Strengthening estimation skills can develop students' understanding of computational procedures.

Benchmarks. One way to estimate is through benchmarks—numbers that serve as reference points for estimating the value of a fraction.[94] The numbers 0, $\frac{1}{2}$, and 1 are useful benchmarks because students generally feel comfortable with them. Students can consider whether a fraction is closest to 0, $\frac{1}{2}$, or 1. For example, when adding $\frac{7}{8}$ and $\frac{3}{7}$, students may reason that $\frac{7}{8}$ is close to 1, and $\frac{3}{7}$ is close to $\frac{1}{2}$, so the answer will be close to $1\frac{1}{2}$.[95] Further, if dividing 5 by $\frac{5}{6}$, students might reason that $\frac{5}{6}$ is close to 1, and 5 divided by 1 is 5, so the solution must be a little more than 5.[96]

Relative Size of Unit Fractions. A useful approach to estimating is for students to consider the size of unit fractions. To do this, students must first understand that the size of a fractional part decreases as the denominator increases.[97] For example, to estimate the answer to $\frac{9}{10} + \frac{1}{8}$, beginning students can be encouraged to reason that $\frac{9}{10}$ is almost 1, that $\frac{1}{8}$ is close to $\frac{1}{10}$, and that therefore the answer will be about 1. More advanced students can be encouraged to reason that $\frac{9}{10}$ is only $\frac{1}{10}$ away from 1, that $\frac{1}{8}$ is slightly larger than $\frac{1}{10}$, and therefore the solution will be slightly more than 1. The principle can and should be generalized beyond unit fractions once it is understood in that context. Key dimensions for generalization include estimating results of operations involving non-unit fractions (e.g., $\frac{3}{4} \div \frac{2}{3}$), improper fractions ($\frac{7}{3} \div \frac{3}{4}$), and decimals ($0.8 \div 0.33$).

Placement of Decimal Point. A common error when multiplying decimals, such as 0.8×0.9 or 2.3×8.7, is to misplace the decimal. Encouraging students to estimate the answer first can reduce such confusion. For example, realizing that 0.8 and 0.9 are both less than 1 but fairly close to it can help students realize that answers such as 0.072 and 7.2 must be incorrect.

3. Address common misconceptions regarding computational procedures with fractions.

Misconceptions about fractions often interfere with understanding computational procedures. The panel believes that it is critical to identify students who are operating with such misconceptions, to discuss the misconceptions with them, and to make clear to the students why the misconceptions lead to incorrect answers and why correct procedures lead to correct answers.

Teachers can present these misconceptions in discussions about how and why some students' computation procedures yield correct answers, whereas others' do not. The group will likely find that many computational errors result from students misapplying rules that are appropriate with whole numbers or with other computational operations with fractions.

Some common misconceptions are described next, together with recommendations for addressing them.

- **Believing that fractions' numerators and denominators can be treated as separate whole numbers.** A common mistake that students make is to add or subtract the numerators and denominators of two fractions (e.g., $\frac{2}{4} + \frac{5}{4} = \frac{7}{8}$ or $\frac{3}{5} - \frac{1}{2} = \frac{2}{3}$).[98] Students who err in this way are misapplying their knowledge of whole number addition and subtraction to fraction problems and failing to recognize that denominators define the size of the fractional part and that numerators represent the number of this part. The fact that this approach is appropriate for multiplication of fractions is another source of support for the misconception.

  Presenting meaningful problems can be useful for overcoming this misconception. For example, a teacher might present the problem, "If you have $\frac{3}{4}$ of an orange and give $\frac{1}{3}$ of it to a friend, what fraction of the original orange do you have left?" Subtracting the numerators and denominators separately would result in an answer of $\frac{2}{1}$ or 2. Students should immediately recognize the impossibility of starting with $\frac{3}{4}$ of an orange, giving some of it away, and ending up with 2 oranges. Such examples can motivate students to think deeply about why treating numerators and denominators as separate whole numbers is inappropriate and can lead them to be more receptive to discussions of appropriate procedures.

- **Failing to find a common denominator when adding or subtracting fractions with unlike denominators.** Students often fail to convert fractions to equivalent forms with a common denominator before adding or subtracting them, and instead just insert the larger denominator in the fractions in the problem as the denominator in the answer (e.g., $\frac{4}{5} + \frac{4}{10} = \frac{8}{10}$).[99] This error occurs when students do not understand that different denominators reflect different-sized unit fractions and that adding and subtracting fractions requires a common unit fraction (i.e., denominator). The same underlying misconception can lead students to make the closely related error of changing the denominator of a fraction without making the corresponding change to the numerator—for example, by converting the problem $\frac{2}{3} + \frac{2}{6}$ into $\frac{2}{6} + \frac{2}{6}$. Visual representations that show equivalent fractions—such as a number line or fraction strip—again can illustrate the need for both common denominators and appropriate changes in numerators.

- **Believing that only whole numbers need to be manipulated in computations with fractions greater than one.** When adding or subtracting mixed numbers, students may ignore the fractional parts and work only with the whole numbers (e.g., $5\frac{3}{5} - 2\frac{1}{7} = 3$).[100] These students are either ignoring the part of the problem they do not understand, misunderstanding the meaning of mixed numbers, or assuming that such problems simply have no solution.[101]

  A related misconception is thinking that whole numbers have the same denominator as a fraction in the problem.[102] This misconception might lead students to translate the problem $4 - \frac{3}{8}$ into $\frac{4}{8} - \frac{3}{8}$ and find an answer of $\frac{1}{8}$. When presented with a mixed number, students with such a misconception might add the whole number to the numerator, as in $3\frac{1}{3} \times \frac{6}{7} = (\frac{3}{3} + \frac{1}{3}) \times \frac{6}{7} = \frac{4}{3} \times \frac{6}{7} = \frac{24}{21}$. Helping students understand the relation between mixed numbers and improper fractions, and how to translate each into the other, is crucial for working with fractions.

- **Treating the denominator the same in fraction addition and multiplication problems.** Students often leave the denominator unchanged on fraction multiplication problems that have equal denominators (e.g., $\frac{2}{3} \times \frac{1}{3} = \frac{2}{3}$).[103] This may occur because students usually encounter more fraction addition problems than fraction multiplication problems; this might lead them to generalize incorrectly

to multiplication the correct procedure for dealing with equal denominators on addition problems. Teachers can address this misconception by explaining the conceptual basis of fraction multiplication using unit fractions (e.g., $\frac{1}{2} \times \frac{1}{2}$ = half of a half = $\frac{1}{4}$). In particular, teachers can show that the problem $\frac{1}{2} \times \frac{1}{2}$ is actually asking what $\frac{1}{2}$ of $\frac{1}{2}$ is, which implies that the product must be smaller than either fraction being multiplied.

- **Failing to understand the invert-and-multiply procedure for solving fraction division problems.** Students often misapply the invert-and-multiply procedure for dividing by a fraction because they lack conceptual understanding of the procedure. One common error is not inverting either fraction; for example, a student may solve the problem $\frac{2}{3} \div \frac{4}{5}$ by multiplying the fractions without inverting $\frac{4}{5}$ (e.g., writing that $\frac{2}{3} \div \frac{4}{5} = \frac{8}{15}$).[104] Other common misapplications of the invert-and-multiply rule are inverting the wrong fraction (e.g., $\frac{2}{3} \div \frac{4}{5} = \frac{3}{2} \times \frac{4}{5}$) or inverting both fractions ($\frac{2}{3} \div \frac{4}{5} = \frac{3}{2} \times \frac{5}{4}$). Such errors generally reflect a lack of conceptual understanding of why the invert-and-multiply procedure produces the correct quotient. The invert-and-multiply procedure translates a multi-step calculation into a more efficient procedure.

The panel suggests that teachers help students understand the multi-step calculation that is the basis for the invert-and-multiply procedure. Teachers can begin by noting that multiplying any number by its reciprocal produces a product of 1, and that dividing any number by 1 leaves the number unchanged. Then teachers can show students that multiplying both fractions by the reciprocal of the divisor is equivalent to using the invert-and-multiply procedure. For the problem $\frac{2}{3} \div \frac{4}{5}$ = (note that we refer to $\frac{2}{3}$ as the dividend and $\frac{4}{5}$ as the divisor):

- multiplying both the dividend ($\frac{2}{3}$) and divisor ($\frac{4}{5}$) by the reciprocal of the divisor yields $(\frac{2}{3} \times \frac{5}{4}) \div (\frac{4}{5} \times \frac{5}{4})$.
- multiplying the original divisor ($\frac{4}{5}$) by its reciprocal ($\frac{5}{4}$) produces a divisor of 1, which results in $\frac{2}{3} \times \frac{5}{4} \div 1$, which yields $\frac{2}{3} \times \frac{5}{4}$.
- thus, the invert and multiply procedure, multiplying $\frac{2}{3} \times \frac{5}{4}$, provides the solution.

4. Present real-world contexts with plausible numbers for problems that involve computing with fractions.

Presenting problems with plausible numbers set in real-world contexts can awaken students' intuitive problem-solving abilities for computing with fractions.[105] The contexts should provide meaning to the fraction quantities involved in a problem and the computational procedure used to solve it. Real-world measuring contexts, such as rulers, ribbons, and measuring tapes, can be useful, as can food—both discrete items (e.g., cartons of eggs, boxes of chocolates) and continuous ones (e.g., pizzas, candy bars).[106] Students themselves can be a helpful source of ideas for relevant contexts, allowing teachers to tailor problems around details that are familiar and meaningful to the students.[107] School events, such as field trips or class parties, track and field days, and ongoing activities in other subjects, also can serve as engaging contexts for problems.

Teachers can help students make connections between a real-world problem and the fraction notation used to represent it. In some cases, students may solve a problem framed in an everyday context but be unable to solve the same problem using formal notation.[108] For instance, they might know that two halves equal a whole but answer the written problem $\frac{1}{2} + \frac{1}{2}$ with $\frac{2}{4}$. Teachers should help students see the connection between the story problem and the fraction

notation and encourage them to apply their intuitive knowledge in both situations. While trying to make connections, teachers can direct students back to the real-world story problem if their students need to ease into understanding the formal notation.[109]

Potential roadblocks and solutions

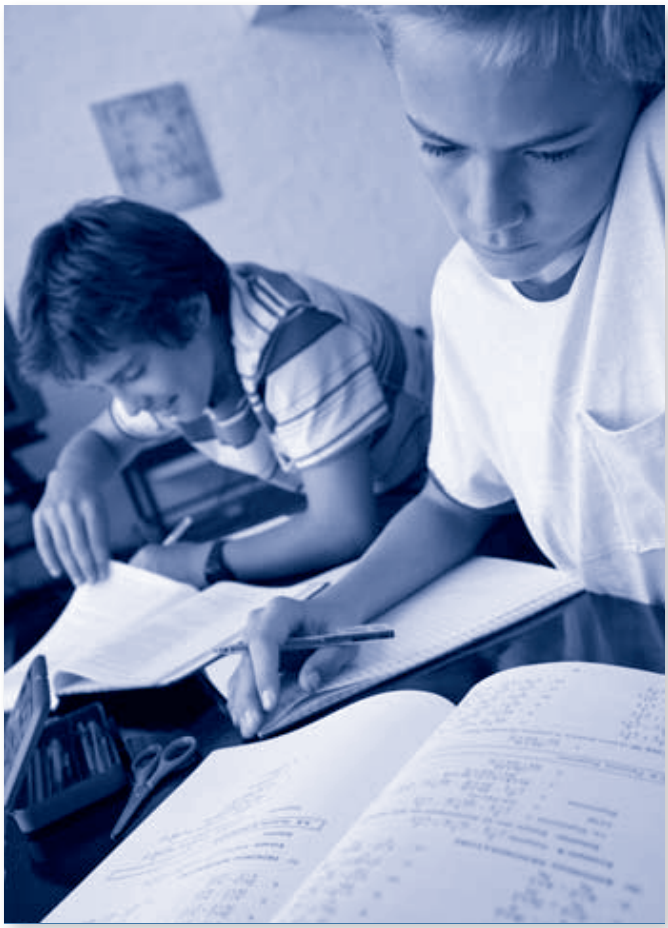
Roadblock 3.1. *Students make computational errors (e.g., adding fractions without finding a common denominator) when using certain pictorial and concrete object representations to solve problems that involve computation with fractions.*

Suggested Approach. Teachers should carefully choose representations that map straightforwardly to the fraction computation being taught. For example, when teaching fraction addition, a representation should demonstrate the need for adding similar units and thus lead students to find a common denominator. Use of some representations can actually reinforce misconceptions. In one study, the use of dot paper for adding fractions led students to *more* often use the incorrect strategy of adding numerators without finding a common denominator.[110] Representations that hold units constant, such as a measuring tape with marked units, can help students see the need for common unit fractions.

Roadblock 3.2. *When encouraged to estimate a solution, students still focus on solving the problem via a computational algorithm rather than estimating it.*

Suggested Approach. Estimation should be presented as a tool for anticipating the size and assessing the reasonableness of an answer. Teachers should focus on the reasoning needed to estimate a solution and should emphasize that estimation is a preliminary step to solving a problem, not a shortcut to obtaining an exact answer. Teachers who pose problems that cannot be solved quickly with mental computation (e.g., problems such as $5/9 + 3/7$ rather than $5/8 + 3/8$) will likely avoid this roadblock.

Develop students' conceptual understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems.



Proportional reasoning is a critical skill for students to develop in preparation for more advanced topics in mathematics.[111] *When students "think proportionally," they understand the multiplicative relation between two quantities.*[112] *For example, understanding the multiplicative relation in the equation* Y = 2X *means understanding that* Y *is twice as large as* X *(and not that* X *is twice as large as* Y*, which is what many students think). Contexts that require understanding of multiplicative relations include problems that involve ratios (i.e., the relation between two quantities, such as the ratio of boys to girls in a classroom), rates (i.e., the relation between two quantities measured in different units, such as distance per unit of time), and proportions (i.e., two equivalent ratios). Proportional reasoning often is needed in everyday contexts, such as adjusting recipes to the number of diners or buying material for home improvement projects; thus proportional reasoning problems provide opportunities to illustrate the value of learning about fractions.*

The panel recommends that teachers develop students' proportional reasoning prior to teaching the cross-multiplication algorithm, using a progression of problems that builds on their informal reasoning strategies. Visual representations are particularly useful for teaching these concepts and for helping students solve problems. After teaching the cross-multiplication algorithm, teachers should return to the informal reasoning strategies, demonstrate that they and the algorithm lead to the same answers on problems for which the informal reasoning strategies are applicable, discuss why they do so, and also discuss problems that can be solved by the cross-multiplication algorithm that cannot easily be solved by the informal strategies.

A caution for teachers: Evidence from many types of problem-solving studies, including ones involving ratio, rate, and proportion, indicates that students often learn a strategy to solve a problem in one context but cannot apply the same strategy in other contexts.[113] *Stated another way, students often do not recognize that problems with different cover stories are the same problem mathematically.*[114] *To address this issue, teachers should point to connections among problems with different cover stories and illustrate how the same strategies can solve them.*

Summary of evidence: Minimal Evidence

Evidence for the recommendation comes from consensus documents that emphasize the importance of proportional reasoning for mathematics learning, as well as the panel's expert opinion.[115] Additionally, the panel separately reviewed evidence relevant to particular action steps within the recommendation. These action steps are supported by case studies demonstrating the variety of strategies students use to solve ratio, rate, and proportion problems; a rigorous study of manipulatives; and two well-designed studies that taught strategies for solving word problems.

Building on developing strategies. Three small case studies provided evidence that students use a variety of strategies to solve proportional reasoning problems (Step 1).[116] Some students initially applied a buildup strategy (e.g., to solve 2:3 = x:12, they added 2:3 four times until they reached 8:12, and then said $x = 8$), whereas others applied a strategy that focused on the multiplicative relation between two ratios (e.g., to solve 2:3 = x:12, they identified the relation between the denominators [$3 \times 4 = 12$] and applied this relation to determine the missing numerator [$2 \times 4 = 8$], then said $x = 8$). However, these studies did not examine whether basing instruction on these strategies improved students' proportional reasoning. The panel believes that students' proportional reasoning can be strengthened through presenting a progression of problems that encourages use of these strategies and that provides a basis for realizing that the cross-multiplication procedure can solve some, but not all, types of problems more efficiently than other strategies.

Using representations. The evidence supporting the use of manipulatives and pictorial representations to teach proportionality concepts is limited (Step 2). However, one study that met WWC standards found that the use of a manipulative improved 4th-graders' ability to visualize and compare two ratios, which improved their ability to solve mixture problems, compared to students who had no exposure to these problems or the manipulative.[117] In another study that met WWC standards, students improved their ability to solve missing value proportion problems by representing information from these problems in a data table that highlighted the multiplicative relationships between quantities.[118] A third well-designed study found a positive impact on student learning of collaboratively constructing pictorial representations relative to using teacher-generated representations.[119] These studies indicate that manipulatives and pictorial representations can be effective teaching tools; however, the principles that determine when they are and are not helpful remain poorly understood.

Teaching problem-solving strategies. The panel also identified limited evidence supporting the recommendation to teach strategies for solving word problems involving ratios and proportions (Step 3). The interventions examined in these studies taught middle school students a four-step strategy for solving ratio and proportion word problems.[120] This strategy developed students' understanding of common problem structures, directed students to use a diagram to identify key information needed to solve a problem, and encouraged students to compare different solution strategies. One of these studies focused on students with learning disabilities, while the other sampled students with a diverse mix of ability levels.[121] Both studies found a positive effect on the accuracy of students' solutions to ratio and proportion problems.

How to carry out the recommendation

1. Develop students' understanding of proportional relations before teaching computational procedures that are conceptually difficult to understand (e.g., cross-multiplication). Build on students' developing strategies for solving ratio, rate, and proportion problems.

Opportunities for students to solve ratio, rate, and proportion problems should be provided prior to teaching the cross-multiplication algorithm.[122] Teachers can use a progression of problems that builds on students' developing strategies for proportional reasoning.[123] In particular, teachers can initially pose problems that allow solutions via the buildup and unit ratio strategies and progress to problems that are easier to solve through cross multiplication. Encouraging students to apply their own strategies, discussing with students varied strategies' strengths and weaknesses, and helping students understand why a problem's solution is correct are advisable.[124] If students do not generate these strategies on their own, teachers should introduce the strategies as ways of solving ratio, rate, and proportion problems.

Teachers can initially pose story problems that allow students to use a buildup strategy, in which they repeatedly add the numbers within one ratio to solve the problem (see Example 4).[125] Problems that facilitate the use of the buildup strategy should have an integral relation between the component numbers in the two ratios—a relation in which the numbers in one ratio can be generated by repeatedly adding numbers in the other ratio, allowing students to build up to the unknown number. For example, the ratios 2:3 and 10:15 have an integral relation, because repeatedly adding 2s and 3s to the first ratio leads to 10:15. Thus, initial problems should involve ratios for which students can easily apply a buildup strategy, such as, "John is baking bread for some friends. He uses 2 cups of flour for every 3 friends. If he wants to make bread for 15 friends, how many cups of flour should he use?"

Next, teachers can present similar problems, but with larger numbers, that demonstrate to students how time-consuming it can be to add up repeatedly to the unknown value. Students will see the advantage of multiplying and dividing rather than depending upon repeated addition. For example, in the baking bread problem, John could be baking bread for all 54 students in the 5th grade.

Teachers can then present problems that cannot be solved immediately either through repeated addition or through multiplying or dividing a given number by a single integer (see Example 4). These are problems that involve ratios without an integral relation, such as $\frac{x}{6} = \frac{3}{9}$. Such problems can be solved by the unit ratio strategy, which involves reducing the known ratio ($\frac{3}{9}$) to a form with a numerator of 1 and then determining the multiplicative relation between the new unit ratio and the ratio with the unknown element ($\frac{x}{6}$). The multiplicative relation between the denominators in the unit ratio and the unknown ratio can then be used to solve for the missing element. For example, $\frac{x}{6} = \frac{3}{9}$ could be solved by expressing $\frac{3}{9}$ as $\frac{1}{3}$, identifying $\frac{2}{2}$ as the number that could be used to multiply $\frac{1}{3}$ and obtain a denominator of 6 without changing the value of $\frac{1}{3}$, multiplying $\frac{1}{3}$ by $\frac{2}{2}$ to obtain $\frac{2}{6}$, and answering "$x = 2$."[126]

The same type of reasoning can be used to solve problems for which the answer is not a whole number; for example, "Susan is making dinner for 6 people and wants to use a recipe that serves 8 people. The recipe for 8 calls for 2 cups of cream. How much cream will she need to serve 6?" This context presents the problem as 2:8 as x:6. Students could solve this problem by reasoning that since 2 cups of cream serve 8 people, 1 cup of cream would serve 4 people, and $1\frac{1}{2}$ cups of cream would serve 6.

Problems such as those in the last paragraph can be used to help students recognize the advantages of a strategy that can solve problems regardless of the particular numbers. Cross-multiplication can be introduced as such an approach. Problems that do not involve integral relations and cannot easily be reduced to unit fractions will help students see the advantages of cross-multiplication, which is essentially a procedure to create equivalent ratios. Its use can be illustrated with problems such as those presented in the previous paragraph that were solved with a unit strategy. For example, students could be encouraged to solve the last problem with the cross-multiplication strategy: writing the equation $2/8 = x/6$ and cross-multiplying to find the missing value. After students arrive at the same answer of $1\frac{1}{2}$, teachers can lead students in a discussion of why the unit ratio and cross-multiplication procedures yield the same answer (see Example 5). Students should practice both with problems that are solved easily through informal reasoning and mental mathematics and with problems that are solved easily using cross-multiplication but not through the buildup or unit ratio strategies. Teachers can encourage students to discuss how to anticipate which approach will be easiest.

Example 4. Problems encouraging specific strategies

Ratio, rate, and proportion problems can be solved using many strategies, with some problems encouraging use of particular strategies. Illustrated below are three commonly used strategies and types of problems on which each strategy is particularly advantageous.

Buildup Strategy

Sample problem. If Steve can purchase 3 baseball cards for $2, how many baseball cards can he purchase with $10?

Solution approach. Students can build up to the unknown quantity by starting with 3 cards for $2, and repeatedly adding 3 more cards and $2, thus obtaining 6 cards for $4, 9 cards for $6, 12 cards for $8, and finally 15 cards for $10.

Unit Ratio Strategy

Sample problem. Yukari bought 6 balloons for $24. How much will it cost to buy 5 balloons?

Solution approach. Students might figure out that if 6 balloons costs $24, then 1 balloon costs $4. This strategy can later be generalized to one in which eliminating all common factors from the numerator and denominator of the known fraction does not result in a unit fraction (e.g., a problem such as $6/15 = x/10$, in which reducing $6/15$ results in $2/5$).

Cross-Multiplication

Sample problem. Luis usually walks the 1.5 miles to his school in 25 minutes. However, today one of the streets on his usual path is being repaired, so he needs to take a 1.7-mile route. If he walks at his usual speed, how much time will it take him to get to his school?

Solution approach. This problem can be solved in two stages. First, because Luis is walking at his "usual speed," students know that $1.5/25 = 1.7/x$. Then, the equation may be most easily solved using cross-multiplication. Multiplying 25 and 1.7 and dividing the product by 1.5 yields the answer of $28\frac{1}{3}$ minutes, or 28 minutes and 20 seconds. It would take Luis 28 minutes and 20 seconds to reach school using the route he took today.

Example 5. Why cross-multiplication works

Teachers can explain why the cross-multiplication procedure works by starting with two equal fractions, such as $\frac{4}{6} = \frac{6}{9}$. The goal is to show that when two equal fractions are converted into fractions with the same denominator, their numerators also are equivalent. The following steps help demonstrate why the procedure works.

Step 1. Start with two equal fractions, for example: $\frac{4}{6} = \frac{6}{9}$.

Step 2. Find a common denominator using each of the two denominators.

a. First, multiply $\frac{4}{6}$ by $\frac{9}{9}$, which is the same as multiplying $\frac{4}{6}$ by 1.

b. Next, multiply $\frac{6}{9}$ by $\frac{6}{6}$, which is the same as multiplying $\frac{6}{9}$ by 1.

Step 3. Calculate the result: $\frac{(4 \times 9)}{(6 \times 9)} = \frac{(6 \times 6)}{(9 \times 6)}$

Step 4. Check that the denominators are equal. If two equal fractions have the same denominator, then the numerators of the two equal fractions must be equal as well, so $4 \times 9 = 6 \times 6$.

Note that in this problem, $4 \times 9 = 6 \times 6$ is an instance of $(a \times d = b \times c)$.

As a result, students can see that the original proportion, $\frac{4}{6} = \frac{6}{9}$, can be solved using cross-multiplication, $4 \times 9 = 6 \times 6$, as a procedure to create equivalent ratios efficiently.

2. Encourage students to use visual representations to solve ratio, rate, and proportion problems.

The panel recommends that teachers encourage the use of visual representations for ratio, rate, and proportion problems. Teachers should carefully select representations that are likely to elicit insight into a particular aspect of ratio, rate, and proportion concepts. For example, a ratio table can be used to represent the relations in a proportion problem (see Figure 9). To identify the amount of flour needed for 32 people when a recipe calls for 1 cup of flour to serve 8, students can use a ratio table to repeatedly add 1 cup of flour per 8 people to find the correct amount for 32 people (i.e., they can use the buildup strategy). Alternatively, students can use the ratio table to see that multiplying the ratio by $\frac{4}{4}$ (i.e., 4 times the recipe) provides the amount of flour needed for 32 people. This visual representation provides a specific referent that teachers can point to as they discuss with students why multiplication leads to the same solution as the buildup strategy.

Figure 9. Ratio table for a proportion problem

Cups of Flour	1	2	3	**4**
Number of People Served	8	16	24	**32**

In addition to using the ratio table as a tool for solving problems, teachers can use it to explore different aspects of proportional relations, such as the multiplicative relations within and between ratios. In the ratio table in Figure 10, the number of cups of flour needed is always 2.5 times the number of people; thus, the ratio between them is always 2.5:1.

As discussed in Recommendation 3, teachers should not always provide representations to students; they sometimes should encourage

them to create their own representations—in this case, representations of ratios, rates, and proportions.[127] Prior to formal instruction in ratios, students tend to use tabular or other systematic forms of record keeping,[128] which can help them understand the functional relation between rows or columns or the numbers in a ratio.[129] Teachers should help students extend these and other representations to a broad range of ratio, rate, and proportion problems.

Figure 10. Ratio table for exploring proportional relations

Cups of Flour	5	7.5	10	12.5
Number of People Served	2	3	4	5

3. Provide opportunities for students to use and discuss alternative strategies for solving ratio, rate, and proportion problems.

The goal is to develop students' ability to identify problems with a common underlying structure and to solve problems that are set in a variety of contexts.[130] Instruction might focus on the meaningful features of different problem types, including ratio and proportion problems, so that students can transfer their learning to new situations. For example, students might first learn to solve recipe problems, such as, "A recipe calls for 3 eggs to make 20 cupcakes. If you want to make 80 cupcakes, how many eggs do you need?" Having learned to solve such problems, students might then be asked to solve similar problems with different contexts, such as: "Building 3 dog-houses requires 42 boards; how many boards are needed to build 9 doghouses?"[131]

Teachers also should help students identify key information needed to solve a problem. Once students can identify the key information in a problem, they can be taught to use diagrams to represent that information.[132] Such diagrams should not simply represent the story problem in diagram form; they also should identify the information needed to solve the problem and the relation between different quantities in the problem. Teachers should encourage students to use different diagrams and strategies to arrive at solutions and should provide opportunities for students to compare and discuss their diagrams and strategies.[133]

The panel suggests using real-life contexts based on students' experiences. A few examples are provided here:[134]

- **Unit price.** Teachers can pose problems based on the unit price of an object, such as comparing the value of two items (e.g., a 16-ounce can of soda for $0.89 and a 12-ounce can of soda for $0.62) and determining how much a certain amount of an item costs given the cost per unit and the number of units purchased. The context of unit-price problems can be buying or selling produce at a grocery store, cans of paint at a hardware store, or any other purchasing situation.

- **Scaling.** Students can solve problems related to the enlargement or reduction of a photo, drawing, or geometric shape (e.g., double the width and double the length of a photo to create a new photo whose area is four times that of the original). Another example of scaling is using a map legend to find the actual distance between two cities, based on their distance on the map.

- **Recipes.** Recipes and cooking provide useful settings for ratio and proportion problems, for example, "If a recipe calls for 1 egg and 3 cups of milk, and the cook wants to make as much as possible using all 8 eggs she has, how much milk is needed, assuming that the ratio of eggs to

milk in the original recipe is maintained?" Students also can revise a recipe to make more or less of the final amount, in situations that call for changing the number of servings or amounts of ingredients using equivalent ratios.

- **Mixture.** Problems related to the mixture of two or more liquids provide another context for posing ratio and proportion problems. Students can compare the concentration of a mixture (e.g., compare the relative amount of one liquid to the amount of another liquid in a mixture) or determine how to maintain the original ratio between liquids in a mixture if the amount of one of the liquids changes.
- **Time/speed/distance.** Students can be told the time, speed, and distance that one car traveled and the values of any two of these variables for a second car and then be asked for the value of the third variable for the second car. For example, they could be told that car A traveled for 2 hours at a rate of 45 miles per hour, so it traveled 90 miles. Then they could be told that car B traveled at the same speed but traveled only 60 miles and be asked to determine the amount of time that car B traveled.

Potential roadblocks and solutions

Roadblock 4.1. *Many students misapply the cross-multiplication strategy.*

Suggested Approach. Carefully presenting several examples of the type shown in Example 5 can help students understand the logic behind the cross-multiplication procedure and why the ratios within the problem need to be in the correct form for the procedure to work. Making sure that students understand the logic of each step in the demonstration takes time, but it can prevent many future errors and misunderstandings.

Roadblock 4.2. *Some students rely nearly exclusively on the cross-multiplication strategy for solving ratio, rate, and proportion problems, failing to recognize that there often are more efficient ways to solve these problems.*

Suggested Approach. Teachers should provide students opportunities to use a variety of strategies for solving ratio, rate, and proportion problems and initially present problems that are easiest to solve with strategies other than cross-multiplication. For example, teachers can present problems in which the relation within the given ratio is integral (e.g., $5/15$) and the relation between the corresponding numbers across the two ratios is not (e.g., $5/15 = 6/x$). These types of problems may encourage students to use prior knowledge of multiplicative relations between numerator and denominator within the ratio where both are known. Requiring students to solve problems mentally (without pencil and paper) also can increase the use of strategies other than cross-multiplication and build number sense with fractions.

Roadblock 4.3. *Students do not generalize strategies across different ratio, rate, and proportion contexts.*

Suggested Approach. In addition to providing students with problems across a variety of contexts and teaching a variety of rate, ratio, and proportion problem-solving strategies, teachers should strive to link new problems with previously solved ones. Teachers can regularly have students judge when the same solution strategy could be used for different types of problems. For example, teachers can demonstrate how information in two types of problems, such as recipes and mixture problems, can be organized in the same way and then compare solution procedures for the two types of problems side by side.

Recommendation 5

Professional development programs should place a high priority on improving teachers' understanding of fractions and of how to teach them.

Teachers play a critical role in helping students understand fraction concepts. Teaching for understanding requires that teachers themselves have a thorough understanding of fraction concepts and operations—including deep knowledge of why computation procedures work. Appropriate use of representations for teaching fractions, a key aspect of the panel's recommendations, requires that teachers understand a range of representations and how to use them to illustrate particular points.

An awareness of common misconceptions and of inappropriate strategies students use to solve fractions problems also is crucial for effective instruction in this area. The panel believes that preservice teacher education and professional development programs must develop teachers' abilities in each of these areas, especially given considerable evidence that many U.S. teachers lack deep understanding of fraction concepts.[135]

Summary of evidence: Minimal Evidence

Despite the limited evidence related to this recommendation, the panel believes teachers must develop their knowledge of fractions and of how to teach them. Researchers have consistently found that teachers lack a deep conceptual understanding of fractions,[136] and that teachers' mathematical content knowledge is positively correlated with students' mathematics achievement.[137] Taken together, these findings suggest a great need for professional development in fraction concepts. Regardless, the evidence rating assigned by the panel recognizes the limited amount of rigorous evidence on the

effects of professional development activities related to fractions.

In one well-designed study, teachers who received training on fraction concepts, on students' understanding of fractions, on students' motivation for learning math, and on how to assess students' knowledge of fractions improved students' conceptual understanding of fractions and their ability to compute with fractions.[138] However, another well-designed study found no impact on student achievement in fractions, decimals, percentages, and proportions, despite offering 7th-grade teachers up to 68 hours of professional development on rational numbers through a summer institute and one-day seminars.[139] Two other studies that met WWC standards provided training on how students develop knowledge and skills related to specific math concepts. One of these studies focused on whole number addition and subtraction and found improvements in students' whole number computation and solutions to word problems. The second study provided teacher training on students' algebraic reasoning and reported a positive impact on student learning.

Research indicates that many elementary school teachers have limited knowledge of fraction concepts and procedures.[140] Interviews with U.S. elementary school teachers showed that a high percentage of them were unable to explain computational procedures for fractions.[141] Another study found that some elementary school teachers had difficulty ordering fractions, adding fractions, and solving ratio problems.[142] Many of the teachers who solved problems correctly could not explain their own problem-solving process. The panel views this limited knowledge of fractions as problematic, given evidence that teachers' mathematical content knowledge is related to students' learning.[143]

How to carry out the recommendation

1. Build teachers' depth of understanding of fractions and computational procedures involving fractions.

To provide effective fractions instruction, teachers need a deep understanding of fraction concepts and operations. In particular, teachers need to understand the reasoning behind computations that involve fractions so they can clearly and coherently explain to students why the procedures work, not just the sequence of steps to take. Without a conceptual understanding of fraction computation, teachers are not likely to help students make sense of fraction operations.[144] Therefore, teacher preparation and professional development activities must support a deeper level of understanding of fractions.[145]

Teachers should have opportunities to gain better understanding of fractions algorithms by solving problems and exploring the meaning of algorithms.[146] One approach is to pose problems that provoke deep discussions of the algorithms, possibly using advanced versions of examples from teachers' lessons.[147] For example, teachers might solve a problem in which they have to equally distribute fractional parts of cake among a number of people (e.g., 3 cakes distributed among 8 people), whereas students might be asked to distribute a whole number of cookies (e.g., 18 cookies among 6 people). Particularly useful are problems or activities that lead teachers to question why an algorithm works or to examine what they do and do not understand about an algorithm.[148] Although teachers can address these problems on their own or in small groups, making time for discussion is crucial.

Having teachers estimate answers to fractions problems and discuss the reasoning that led to the estimates also can be useful. All

activities should eventually link back to the classroom, with opportunities for teachers to discuss how they would respond to students' questions about why estimation is valuable and the logic that separates effective and less effective estimation procedures.

Professional development should not focus exclusively on fraction topics covered at the teacher's grade level. Teachers must understand fraction concepts covered in the entire elementary and middle school curricula and should know how these concepts fit within the broader math curriculum. Awareness of fraction concepts taught in earlier grades ensures that teachers can build on what students already know; it also can help teachers identify and address common misconceptions that students might have developed. Understanding fraction concepts and other more advanced mathematics that will be covered in later grades helps teachers set goals and think about how their teaching can provide foundations for ideas that students will encounter in the future.

2. Prepare teachers to use varied pictorial and concrete representations of fractions and fraction operations.

To use concrete and pictorial representations effectively, teachers must understand how these representations link to fraction concepts and how they can be used to improve student learning. Teacher education and professional development activities should prepare teachers to use such representations for teaching fractions and should help teachers understand how the representations relate to the concepts being taught.

Teachers might learn, for example, that diagrams of sharing scenarios can help highlight the link between fractions and division (i.e., the quotient interpretation of fractions) by allowing students to represent fractions with equal shares (e.g., 2 large brownies shared among 5 children). Number lines can focus students on measurement interpretations of fractions, with fractions representing a distance between two numbers. Area models— particularly rectangular ones, but models using other shapes as well—can be used to depict part-whole representations of fractions.

Development activities should provide opportunities for teachers to integrate representations into fractions lessons.[149] In addition, teachers need to understand difficulties that might arise when they use a pictorial or concrete representation to teach fractions. For example, students may view the entire number line, rather than the distance between two numbers, as the unit when locating fractions (e.g., they might interpret the task of locating ¾ on a 0-to-5 number line as locating the point 75% of the way across the number line). Professional development activities need to help teachers anticipate misconceptions and learning problems that are likely to arise, and identify ways of addressing them.

3. Develop teachers' ability to assess students' understandings and misunderstandings of fractions.

Professional development activities with teachers should emphasize how students develop an understanding of fractions and the obstacles students face in learning about them.[150] Information from research on the development of fraction learning should be provided in these discussions.[151]

One method that is useful for meeting this goal is to provide teachers with opportunities to analyze and critique student thinking about fractions. This can be done by examining students' written work or watching video clips of students solving problems that are designed to provide insight into

students' thinking.[152] For example, teachers can be asked to analyze sources of students' difficulty on problems such as, "Paige had 3 boxes of cereal. Each box was ⅔ full. If the cereal in the 3 boxes were poured into empty boxes, how many boxes would it fill? Use rectangular drawings or a number line to display your reasoning." Teachers can be asked to video-record students' performance on such problems before a professional development session; then teachers can bring students' work or video clips to the session and use them as a basis for discussion.

Teachers should know the types of mistakes students most often make when working with fractions and also should understand the underlying misconceptions that cause them. Analyzing students' work is a useful way to identify problem areas and to gain insight into students' thought processes. To be most effective, teachers must know how to design problems that diagnose the source of errors. For example, teachers might structure a decimal-ordering problem to assess whether students understand place value (e.g., ordering the following decimals from smallest to largest: 0.2, 0.12, 0.056).

Preservice and in-service activities should help teachers understand research on children's knowledge of fractions; the research presented should be chosen to inform teachers' assessment activities and instruction. For example, research has shown that students often have difficulty with fraction names and with understanding the value of fractions.[153] Whether students in a given classroom are having such difficulty can be assessed by asking them to state fractions that label the locations of hatch marks on a number line with endpoints of 0 and 1. Such an assessment might indicate that students refer to a variety of locations as ½, or that they view fractions with larger denominators as larger than fractions with smaller denominators (e.g., they might think that $\frac{1}{8} > \frac{1}{3}$). Such a pattern might lead to an engaging and productive discussion of how the system for naming fractions works and why that naming procedure makes sense. More generally, development activities should provide opportunities for teachers to practice writing or selecting problems that accurately assess students' understanding and to use assessment results to design useful lessons.

Potential roadblocks and solutions

Roadblock 5.1. *Administrators or professional development personnel might argue that the topic of fractions is just one of many that elementary and middle school teachers must be prepared to teach and that their district, program, or school cannot devote more time or resources to it.*

Suggested Approach. The panel recognizes that time and resources for providing professional development are limited. However, a convincing argument can be made for devoting some time and resources to this topic: (1) fractions are a critical foundation for more advanced mathematics, (2) many teachers lack sufficient understanding of fraction to teach the topic effectively, and (3) U.S. students lag further behind those in other countries in solving problems with fractions than in solving problems with whole numbers.[154] The panel believes the need is critical for elementary and middle school teachers to receive professional development related to their content knowledge of fractions and to the teaching of fractions, including decimals, percentages, ratios, rates, and proportions. The panel suggests that school and district leaders consider fractions a high priority for professional development.

Roadblock 5.2. *Some teachers have difficulty with whole number topics, such as multiplication and division, that are related to the teaching of fractions.*

Suggested Approach. A deep understanding of whole number multiplication and division, including why and how common computational algorithms work, is essential for teaching fractions effectively. When selecting or designing professional development activities related to fractions, education leaders should consider whether reviewing these key whole number topics is a necessary prerequisite for teachers in the particular school or district.

Roadblock 5.3. *Some teachers do not think additional professional development involving fractions is necessary.*

Suggested Approach. Although most teachers are able to compute with fractions, many do not have a strong conceptual background regarding fractions or an understanding of the logic underlying computational algorithms used for solving fraction problems. By first determining if teachers know why and how common computational algorithms (e.g., invert and multiply) work and why certain steps within algorithms are necessary (e.g., establishing common denominators for addition and subtraction), education leaders can decide whether professional development involving fractions is an important need in their schools or districts.

C

Common fraction – A fraction written in the form a/b, where both a and b are integers and b does not equal zero (e.g., $\frac{3}{4}$, $\frac{6}{5}$, $-(\frac{1}{8})$).

Covariation – A measure of how much two quantities change together. For example, the extent to which one quantity increases as another quantity increases.

D

Denominator – For any fraction a/b, the denominator is the number below the hash line. The denominator represents the divisor of a division problem, or the number of parts into which a whole amount is divided (e.g., for the fraction $\frac{2}{3}$, the denominator 3 refers to a whole divided into three parts).

E

Equal sharing – The activity of completely distributing an object or set of objects equally among a group of people.

Equivalent fractions – Fractions that represent the same numerical value; equal fractions. For example, $\frac{2}{4}$ and $\frac{4}{8}$ are both equal to $\frac{1}{2}$; therefore $\frac{2}{4}$, $\frac{4}{8}$, and $\frac{1}{2}$ are equivalent fractions.

F

Fraction density – The concept that between any two fractions there is another fraction. For example, the fraction $\frac{1}{4}$ is between 0 and $\frac{1}{2}$; the fraction $\frac{1}{8}$ is between 0 and $\frac{1}{4}$; and the fraction $\frac{1}{16}$ is between 0 and $\frac{1}{8}$. One consequence of this fact is that between any two fractions there are an infinite number of fractions.

I

Improper fraction – A fraction with a numerator that is greater than or equal to the denominator. Examples of improper fractions include $\frac{5}{5}$, $\frac{9}{8}$, and $\frac{14}{9}$.

M

Mixed number – A fraction written as a whole number and a fraction less than one. Examples of mixed numbers include $1\frac{2}{3}$, $4\frac{3}{8}$, and $-2\frac{5}{6}$.

Multiplicative relation – A relation between two quantities in which one quantity can be multiplied by a factor to obtain a second quantity.

N

Numerator – For any common fraction a/b, the numerator is the number above the hash line. The numerator represents the dividend of a division problem or the number of fractional parts represented by a fraction (e.g., for the fraction $\frac{2}{3}$, the numerator 2 represents the number of thirds).

P

Percent – Any number expressed as a fraction or ratio of 100 (i.e., with a denominator of 100). For example, 75% is equivalent to 0.75 or $^{75}/_{100}$.

Proportion – An expression of two equivalent ratios or fractions. A proportion is an equation written in the form $^{a}/_{b} = {}^{c}/_{d}$, thus indicating that the two ratios are equivalent.

Proportional reasoning – The literature consists of several different definitions of proportional reasoning. On a basic level, the term means understanding and working with the underlying relations in proportions.[155] Others describe proportional reasoning as the ability to compare one relative amount to another,[156] or the ability to understand multiplicative relations or reason about multiplicative situations.[157]

Q

Quotient – The solution to a division problem. For example, 3 is the quotient for the following division problem: 12 ÷ 4 = 3.

R

Rational number – Any number that can be expressed in the form $^{a}/_{b}$ where a and b are both integers and b does not equal zero. Rational numbers can take many different forms, including common fractions, ratios, decimals, and percents.

Rate – The relation between two quantities measured in different units. For example, distance per unit of time.

Ratio – The relation between two quantities. For example, the ratio 2:3 might represent the relationship of the number of boys to girls in a classroom, or two boys for every three girls in the class.

U

Unit fraction – A fraction with a numerator of one (e.g., $^{1}/_{3}$, $^{1}/_{11}$).

Unit ratio – A ratio with a denominator of one (e.g., 5:1, 9:1).

W

Whole numbers – The set of numbers starting with zero and increasing by one (i.e., 0, 1, 2, 3…).

Appendix A

Postscript from the Institute of Education Sciences

What is a practice guide?

The Institute of Education Sciences (IES) publishes practice guides to share rigorous evidence and expert guidance on addressing education-related challenges not solved with a single program, policy, or practice. Each practice guide's panel of experts develops recommendations for a coherent approach to a multifaceted problem. Each recommendation is explicitly connected to supporting evidence. Using standards for rigorous research, the supporting evidence is rated to reflect how well the research demonstrates that the recommended practices are effective. Strong evidence means positive findings are demonstrated in multiple well-designed, well-executed studies, leaving little or no doubt that the positive effects are caused by the recommended practice. Moderate evidence means that well-designed studies show positive impacts, but some questions remain about whether the findings can be generalized or whether the studies definitively show that the practice is effective. Minimal evidence means data may suggest a relationship between the recommended practice and positive outcomes, but research has not demonstrated that the practice is the cause of positive outcomes. (See Table 1 for more details on levels of evidence.)

How are practice guides developed?

To produce a practice guide, IES first selects a topic. Topic selection is informed by inquires and requests to the What Works Clearinghouse Help Desk, formal surveys of practitioners, and a limited literature search of the topic's research base. Next, IES recruits a panel chair who has a national reputation and expertise in the topic. The chair, working with IES, then selects panelists to co-author the guide. Panelists are selected based on their expertise in the topic area and the belief that they can work together to develop relevant, evidence-based recommendations. IES recommends that the panel include at least one practitioner with relevant experience.

The panel receives a general template for developing a practice guide, as well as examples of published practice guides. Panelists identify the most important research with respect to their recommendations and augment this literature with a search of recent publications to ensure that supporting evidence is current. The search is designed to find all studies assessing the effectiveness of a particular program or practice. These studies are then reviewed against the What Works Clearinghouse (WWC) standards by certified reviewers who rate each effectiveness study. WWC staff assist the panelists in compiling and summarizing the research and in producing the practice guide.

IES practice guides are then subjected to rigorous external peer review. This review is done independently of the IES staff that supported the development of the guide. A critical task of the peer reviewers of a practice guide is to determine whether the evidence cited in support of particular recommendations is up-to-date and that studies of similar or better quality that point in a different direction have not been overlooked. Peer reviewers also evaluate whether the level of evidence category assigned to each recommendation is appropriate. After the review, a practice guide is revised to meet any concerns of the reviewers and to gain the approval of the standards and review staff at IES.

A final note about IES practice guides

In policy and other arenas, expert panels typically try to build a consensus, forging statements that all its members endorse. But practice guides do more than find common ground; they create a list of actionable recommendations. When research clearly shows which practices are effective, the panelists use this evidence to guide their recommendations. However, in some cases, research does not provide a clear indication

of what works, and panelists' interpretation of the existing (but incomplete) evidence plays an important role in guiding the recommendations. As a result, it is possible that two teams of recognized experts working independently to produce a practice guide on the same topic would come to very different conclusions. Those who use the guides should recognize that the recommendations represent, in effect, the advice of consultants. However, the advice might be better than what a school or district could obtain on its own. Practice guide authors are nationally recognized experts who collectively endorse the recommendations, justify their choices with supporting evidence, and face rigorous independent peer review of their conclusions. Schools and districts would likely not find such a comprehensive approach when seeking the advice of individual consultants.

Institute of Education Sciences

Appendix B

About the Authors

Panel

Robert Siegler, Ph.D., is the Teresa Heinz Professor of Cognitive Psychology at Carnegie Mellon University. His current research focuses on the development of estimation skills and how children's basic understanding of numbers influences their estimation and overall math achievement. The general overlapping waves theory of cognitive development, described by Siegler in his 1996 book *Emerging Minds*, has proven useful for understanding the acquisition of a variety of math skills and concepts. His other books include *How Children Discover New Strategies*, *How Children Develop*, and *Children's Thinking*. Siegler was the Tisch Distinguished Visiting Professor at Teachers College, Columbia University, for 2009/10 and was a member of the President's National Mathematics Advisory Panel from 2006 to 2008. He received the Brotherton Fellowship from the University of Melbourne in 2006 and the American Psychological Association's Distinguished Scientific Contribution Award in 2005.

Thomas Carpenter, Ph.D., is emeritus professor of curriculum and instruction at the University of Wisconsin–Madison and director of the Diversity in Mathematics Education Center for Learning and Teaching. His research investigates how children's mathematical thinking develops, how teachers use specific knowledge about children's mathematical thinking in instruction, and how children's thinking can be used as a basis for professional development. Dr. Carpenter is a former editor of the *Journal for Research in Mathematics Education*. Along with Elizabeth Fennema, Megan Franke, and others, he developed the Cognitively Guided Instruction research and professional development project. He is currently focused on issues of equity and social justice in mathematics teaching and learning.

Francis (Skip) Fennell, Ph.D., is a professor of education at McDaniel College and project director of the Elementary Mathematics Specialists and Teacher Leaders Project. Dr. Fennell is widely published in professional journals and textbooks related to elementary and middle grade mathematics education, and he served on the National Mathematics Advisory Panel, chairing the Conceptual Knowledge and Skills Task Group. In 2008, he completed a two-year presidency of the National Council of Teachers of Mathematics. He served as one of the writers of the *Principles and Standards for School Mathematics* and the *Curriculum Focal Points*, both for the National Council of Teachers of Mathematics. Dr. Fennell is currently principal investigator on a project aimed at designing a graduate curriculum for elementary mathematics teacher leaders, creating a clearinghouse of existing materials, and developing support materials for elementary mathematics specialists and teacher leaders.

David Geary, Ph.D., is a cognitive-developmental psychologist, as well as a Curators' Professor and Thomas Jefferson Professor in the department of psychological sciences at the University of Missouri. He has been studying developmental and individual differences in basic mathematical competencies for more than 20 years and is currently directing a longitudinal study of children's mathematical development and learning disorders. Geary is the author of three books, including *Children's Mathematical Development*, and the co-author of a fourth. In addition, he served on the National Mathematics Advisory Panel and was one of the primary contributors to the 1999 Mathematics Framework for California Public Schools for kindergarten through grade 12. Distinctions received include the Chancellor's Award for Outstanding Research and Creative Activity in the Social and Behavioral Sciences and a MERIT award from the National Institutes of Health.

W. James (Jim) Lewis, Ph.D., is the Aaron Douglas Professor of Mathematics at the University of Nebraska–Lincoln (UNL), as well as director of the school's Center for Science, Mathematics, and Computer Education. Dr. Lewis is principal investigator for two

National Science Foundation Math Science Partnerships, NebraskaMATH and the Math in the Middle Institute Partnership. He was chair of the Conference Board of the Mathematical Sciences committee that produced *The Mathematical Education of Teachers* and co-chair of the National Research Council committee that produced the report *Educating Teachers of Science, Mathematics, and Technology: New Practices for the New Millennium*. Dr. Lewis was also co-principal investigator for Math Matters, a National Science Foundation grant to revise the mathematics education of future elementary school teachers at UNL.

Yukari Okamoto, Ph.D., is a professor in the department of education at the University of California–Santa Barbara. Her work focuses on cognitive development, the teaching and learning of mathematics and science, and cross-cultural studies. She is particularly interested in children's acquisition of mathematical, scientific, and spatial concepts and participated in the video studies of mathematics and science teaching as part of the Third International Mathematics and Science Study (TIMSS). Dr. Okamoto's recent publications include *Fourth-Graders' Linking of Rational Number Representation: A Mixed Method Approach* and *Comparing U.S. and Japanese Elementary School Teachers' Facility for Linking Rational Number Representations*.

Laurie Thompson, M.A., has 10 years of experience as an elementary teacher and math resource teacher. Her experience includes teaching 1st, 3rd, 4th, and 5th grades in Carroll County, Maryland; Loudon County Public Schools, Virginia; and Katy Independent School District, Texas. As an elementary math resource teacher in Loudon County, Ms. Thompson worked with elementary math teachers to team-teach lessons, organize guided instructional centers, and conduct small-group instruction. In this role, she also developed and evaluated mathematics lessons and materials for kindergarten through 5th-grade classrooms. She has served as a mentor and team leader for new teachers and participated in professional learning communities.

Jonathan Wray, M.A., is the instructional facilitator for secondary mathematics curricular programs in the Howard County (Maryland) Public School System. He recently completed a two-year term as president of the Maryland Council of Teachers of Mathematics (MCTM). Mr. Wray was selected as the MCTM Outstanding Teacher Mentor in 2002 and as his district's Outstanding Technology Leader in Education by the Maryland Society for Educational Technology in 2004. He serves on the editorial panel of *Teaching Children Mathematics*, a peer-reviewed journal produced by the National Council of Teachers of Mathematics. Mr. Wray also has served as a classroom teacher for primary and intermediate grades, a gifted/talented resource teacher, an elementary mathematics resource teacher, a curriculum and assessment developer, and an educational consultant.

Staff

Jeffrey Max is a researcher at Mathematica Policy Research with experience conducting evaluations in the education area. His current work focuses on teacher quality issues, including measures of teacher effectiveness, the distribution of teacher quality, and teacher-compensation reform. Mr. Max also contributes to the What Works Clearinghouse, previously working on the practice guide that addresses access to higher education. His prior experience includes teaching kindergarten in a New Orleans public school.

Moira McCullough is a research analyst at Mathematica Policy Research and has experience with education evaluations and research. Ms. McCullough has worked extensively for the What Works Clearinghouse. She is a certified reviewer of studies across several areas and coordinated the elementary school math topic area. She contributed to the practice

guide addressing access to higher education and to research perspectives synthesizing expert recommendations to states and school districts for use of funds from the American Recovery and Reinvestment Act. She also has experience with measures of teacher effectiveness in mathematics.

Andrew Gothro is a research analyst at Mathematica Policy Research. He has experience providing research support and conducting quantitative data analysis on topics ranging from child development to antipoverty programs. Mr. Gothro supported the panel on this project by identifying and organizing relevant research, synthesizing findings from reviewed studies, and crafting language for an audience of education practitioners.

Sarah Prenovitz is a research assistant/programmer at Mathematica Policy Research. She has experience providing research support and conducting data analysis for studies of teacher incentive programs and professional development programs, as well as programs to support and encourage employment for persons with disabilities. She also has developed companion materials to accompany a curriculum for Head Start staff on using continuous assessment to shape classroom practice.

Appendix C

Disclosure of Potential Conflicts of Interest

Practice guide panels are composed of individuals who are nationally recognized experts on the topics about which they are making recommendations. IES expects the experts to be involved professionally in a variety of matters that relate to their work as a panel. Panel members are asked to disclose these professional activities and institute deliberative processes that encourage critical examination of their views as they relate to the content of the practice guide. The potential influence of the panel members' professional activities is further muted by the requirement that they ground their recommendations in evidence that is documented in the practice guide. In addition, before all practice guides are published, they undergo an independent external peer review focusing on whether the evidence related to the recommendations in the guide has been presented appropriately.

The professional activities reported by each panel member that appear to be most closely associated with the panel recommendations are noted below.

Jim Lewis receives royalties as an author of *Math Vantage*, a mathematics curriculum for middle school students. This program is not mentioned in the guide.

Appendix D

Rationale for Evidence Ratings[a]

The panel conducted an initial search for research from 1989 to 2009 on practices for improving students' learning of fractions. The search focused on studies of interventions for teaching fractions to students in kindergarten through 8th grade that did not exclusively focus on students with diagnosed learning disabilities; studies examined students in both the United States and other countries.

Panelists identified more than 3,000 studies through this initial search, including 125 with causal designs reviewed according to What Works Clearinghouse (WWC) standards. Twenty-six of the studies met evidence standards with or without reservations. Given the limited research on practices for improving students' fraction knowledge, the panel expanded its search beyond fractions to identify studies relevant for number lines (Recommendation 2) and professional development (Recommendation 5). This led panel members to an additional seven studies that met standards with or without reservations. Of the 33 total studies that met the causal validity standards of the WWC and were broadly related to the topic, 20 studies were relevant to the panel's specific recommendations. The panel also examined studies that did not have designs eligible for a WWC review but were relevant to the recommendations, including correlational studies, case studies, and teaching experiments.

Recommendation 1. Build on students' informal understanding of sharing and proportionality to develop initial fraction concepts.

Level of evidence: Minimal Evidence

The panel assigned a rating of *minimal evidence* to this recommendation. The recommendation is based on seven studies showing that students have an early understanding of sharing and proportionality[158] and two studies of instruction that used sharing scenarios to teach fraction concepts.[159] However, none of the studies in this latter group met WWC standards. Despite this limited evidence, the panel believes that students' informal knowledge of sharing and proportionality provides a foundation for teaching fraction concepts.

The panel separately examined the research on sharing activities and proportional relations for this recommendation.

Sharing activities. Children have the ability to create equal shares at an early age. Children as young as age 5 can complete basic sharing tasks that involve evenly distributing a set of 12 or 24 objects among two to four recipients.[160] In one study, most 5-year-old children could do this even when using different-size units (i.e., equally distributing single, double, and triple blocks). The ability to create equal shares improves with age, with 6-year-old children performing better than 4- and 5-year-olds.[161] Sharing continuous objects is more difficult for young children than sharing a set of objects: children in one study had more difficulty sharing a rope among three recipients than sharing a set of crackers.[162]

Children's understanding of how to share does not necessarily extend to underlying fraction concepts. Many students do not understand that sharing the same set of objects with more people results in smaller shares for each person.[163] One study that potentially met standards showed an improved understanding of this concept among kindergarten students who were given results from sharing scenarios with different numbers of sharers (i.e., different denominators).[164] This study demonstrated the potential for using sharing activities as the basis for teaching early fraction concepts. However, a review of the study could not be completed because insufficient information was provided on the number of schools assigned to each condition.

Two case studies showed how an early understanding of sharing could be used to teach fractions to elementary students.[165]

[a] Eligible studies that meet WWC evidence standards or meet evidence standards with reservations are indicated by **bold text** in the endnotes and references pages.

In both studies, a teacher posed various story problems based on sharing scenarios to teach fraction concepts such as equivalence and ordering, as well as fraction computation. For example, students solved problems about people sharing varying amounts of food or the use of seating arrangements to share a set of objects in different ways. The instruction in both studies included story problems based on realistic situations, opportunities for students to use their own drawings and strategies to obtain solutions, and whole-class discussions.

One of these studies examined a Dutch curriculum for 4th-grade students but did not meet standards because only one classroom was assigned to the treatment.[166] The other study presented a five-week instructional unit to 17 1st-grade students but lacked a control group, so it did not have a reviewable design. However, both studies reported positive results with using sharing scenarios to teach fraction concepts.[167]

Proportional relations. Young children have an early understanding of proportional relations. Three studies presented proportions with geometric figures or everyday objects and had students identify or create a matching proportion.[168] For example, in one study, the experimenter pretended to eat a portion of a pizza and had children pretend to eat the same proportion from a box of chocolates.[169] In another study, students matched proportions represented by boxes filled with blue and white bricks.[170] By age 6, children matched equivalent proportions in all of these studies.

One-half plays an important role in children's early proportional reasoning abilities. Children performed better when choosing between options that crossed the half point (i.e., one figure was more than one-half filled, and the other was less than one-half filled) than between two proportions that were both more than or less than one-half filled.[171] Children tended to have more difficulty matching proportions represented by discrete objects than by continuous objects.[172]

Children's early understanding of proportional relations also is reflected in their ability to solve basic analogies. Analogies are similar to proportions in that students must identify a relation in the first set of items and then apply this relation to a second set of items. One study found that children ages 6 and 7 performed above chance on analogies based on simple patterns or proportional equivalence.[173] For example, students could complete the analogy, "Half circle is to half rectangle as quarter circle is to quarter rectangle." One study found that children could map the relative sizes of items within a three-item set to the relative sizes of items within another set of three objects.[174] For example, when the experimenter selected the largest of three different-size cups, children could pick the corresponding cup from their set of three cups.

The panel did not identify studies meeting standards that examined the effect of using this early knowledge to teach fraction concepts. However, one study that potentially met standards examined a way to improve students' ability to match equivalent proportions.[175] The author provided 6- to 8-year-old children with feedback and explanations about how to use the half boundary to identify equivalent proportions. This strategy focused children on the part-part relation between shaded and unshaded areas used to represent proportions; the author reported positive effects on children's ability to identify which of two glasses was more full—and, therefore, on whether students could differentiate between absolute and relative amounts of water.

Recommendation 2.
Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

Level of evidence: Moderate Evidence

The panel rates this recommendation as being supported by *moderate evidence*, based on

three studies that met WWC standards and used number lines to teach students about the magnitudes of whole numbers;[176] one study that met WWC standards and showed that instruction with number lines improved students' understanding of decimal fractions;[177] and four studies that showed strong correlations between number line estimates with whole numbers and performance on arithmetic and mathematical achievement tests.[178] Another study demonstrated that a property of number line estimates that has been documented extensively with whole numbers also is present with fractions (specifically, logarithmic to linear transitions in patterns). This suggests that representations of numerical magnitudes influence understanding of fractions as well as of whole numbers.[179] The panel believes that given the clear applicability of number lines to fractions as well as whole numbers, these findings indicate that number lines can improve fraction learning for elementary and middle school students.

The evidence to support this recommendation includes studies that examined the use of number lines and other linear representations to teach whole number and fraction concepts.

Number lines for whole number concepts. (see Table D.1.) Three studies that met standards found that briefly playing a linear board game with numbers improved preschool students' understanding of whole number magnitude.[180] In the studies, students from low-income backgrounds played a numeric board game 20 to 30 times over the course of four to five sessions lasting 15 to 20 minutes each. The game involved moving a marker one or two spaces at a time across a horizontal board that had the numbers 1 to 10 listed in order from left to right in consecutive squares. Students used a spinner to determine whether to make one or two moves and then said out loud the number they had spun and the numbers on the squares as they moved. The experimenter played the game with each child and helped each correctly name numbers. Control students in two of the studies played the same game but with colors rather than numbers,[181] and control students in the other study completed counting and number-identification tasks.[182]

The linear board game, which the panel views as a proxy for number lines, had a positive effect on students' ability to compare the size of whole numbers. Authors of the three studies reported significant effect sizes of 0.75, 0.99, and 1.17 on accuracy in comparing whole numbers (from 0 to 10).[183] The linear board game also improved participating students' ability to locate whole numbers on a number line accurately. These studies measure the accuracy of students' number line estimates using a measure called percent absolute error, which is the difference between a student's estimate and the actual number divided by the scale of the number line. Two of the studies found effect sizes for percent absolute error of 0.63 (author reported) and 0.86 (WWC calculated).[184] One of the studies also reported that playing the game significantly improved students' ability to learn the answers to addition problems on which they received feedback.[185]

Research supporting the use of number lines with whole numbers includes two additional studies that met WWC standards. One of the studies had students place 10 evenly spaced numbers on a number line before locating numbers on a 0-to-100 number line.[186] The authors report that this approach led to a substantively important but not significant increase in the accuracy of students' number line estimates, whereas students in the control group, who located one number at a time, did not improve.[187] (The WWC defines substantively important, or large, effects on outcomes to be those with effect sizes greater than 0.25 standard deviations.[188])

The second supporting study used a number line to improve 1st-grade students' performance on addition problems for which they had been trained.[189] Treatment group students viewed the addends and sums of four addition problems on a number line; control group students received feedback on the problems

Table D.1. Studies of interventions that used number lines to improve understanding of whole number magnitude that met WWC standards (with or without reservations)

Citation	Grade Level	Analysis Sample Size	Intervention	Comparison
Siegler and Ramani (2008)	Preschool	36 students	Students play a linear board game with numbers.	Students play a linear board game with colors.
Ramani and Siegler (2008)	Preschool	112 students	Students play a linear board game with numbers.	Students play a linear board game with colors.
Siegler and Ramani (2009)	Preschool	88 students	Students play a linear board game with numbers.[190]	Students participate in counting and number-identification tasks.
Siegler and Booth (2004)	1st and 2nd[191]	55 students	Students place 10 evenly spaced numbers on a number line.	Students use number lines without locating evenly spaced numbers first.
Booth and Siegler (2008)	1st	52 students	Students receive a number line showing addends and sums for trained addition problems.[192]	Students solve trained addition problems without a number line.

but did not use a number line. The authors reported that treatment group students were more likely than control group students to answer the same addition problems correctly later. In addition, the study noted that the number line experience led to improved quality of errors on the addition problems (errors that were closer to the correct answer).

The panel also identified evidence showing a relation between students' accuracy in locating whole numbers on a number line and general math achievement.[193] These studies show a positive significant relation between the linearity of number line estimates and general math achievement for students in kindergarten through 4th grade, with correlations ranging from 0.39 to 0.69. The accuracy of number line estimates (i.e., how close a number is to its actual position) was positively related to general math achievement, with one study finding a significant relation ranging from 0.37 to 0.66);[194] an additional study found positive but non-significant relations for 1st- and 2nd-graders in one experiment and significant positive relations for 2nd- and 4th-graders in another experiment.[195]

Number lines for teaching fraction concepts. One study that met WWC standards examined the use of number lines for comparing the magnitude of decimals.[196] Sixty-one students in 5th and 6th grades played a computer game in which they located a decimal's position on a 0-to-1 number line. Students in the treatment and control groups completed 15 problems during sessions lasting about 40 minutes. The study involved three treatment groups that received interventions designed to help students correctly represent the problem: the first treatment group received a prompt for students to notice the tenths digit of each decimal, the second group used a number line with the tenths place marked, and the third group received both the prompts and marked tenths on the number line. Students in the control group also solved computer-based number line problems, but without the assistance of these interventions.

Since students in both the treatment and control groups used number lines, the study does not provide causal evidence for whether using number lines improves students' understanding of decimals. However, the results indicate that focusing on certain aspects of the number line—specifically, noticing and marking the tenths place—led to significant improvements in students' ability to locate decimals

on a number line. The combination of placing tenths markings on the number lines and prompting students to notice and think about them (treatment group 3) significantly improved students' ability to locate decimal fractions on a number line relative to when neither was present (effect size of 0.57). When students only received the tenths markings (treatment group 1) or only heard the prompts (treatment group 2), the interventions did not have a significant effect. The panel believes this outcome indicates that the combination of the prompts and markings, together with use of the number line, leads to increased understanding of decimals' magnitude.

A comparison of students' conceptual understanding of decimals before and after the intervention provides additional evidence on the usefulness of number lines. Playing the computer-based number line game led to improvements in treatment and control students' conceptual understanding of decimals, including their ability to compare relative magnitudes of fractions, identify equivalent fractions, and understand place value. This is suggestive evidence, because there is no comparison group of students who did not use a number line.

Another study examined the use of number lines in fractions instruction but did not meet standards.[197] The study compared two Dutch curricula over the course of a school year. One curriculum focused on the use of number lines and measurement contexts to teach fractions; the other curriculum used circles and part-whole representations of fractions. Students in the treatment group measured objects using different-size bars and compared fractions on a number line. The authors reported positive effects on 9- to 10-year-olds' understanding of fractions. However, the study did not meet standards, because only one classroom of students was assigned to the treatment. Another problem in interpreting the study was that the experimental group encouraged student interaction, whereas the control group students primarily worked alone. As a result, distinguishing the effect of these instructional approaches from the effect of the curriculum was not possible.

Two additional studies that were not eligible for review found mixed results of using a number line to teach fraction concepts.[198] One study examined using a number line to teach fraction addition to a class of 6th-grade students. Based on classroom observations and interviews with the teacher and two students, the authors found that students had difficulty viewing partitions on a number line as fixed units, as well as difficulty associating equivalent fractions with a single point on the number line. Minor differences in how the teacher presented the number line affected whether students viewed the partitions as fixed units.

The second study described three small case studies of fraction instruction that used number lines for representing and ordering fractions.[199] In this study, 4th- and 5th-grade students had trouble locating fractions on a number line when fractions were in reduced form and the number line was organized by a smaller unit fraction (e.g., they had difficulty locating ⅓ on a number line divided into sixths). However, the authors also reported that number line instruction improved students' ability to work with fractions.

Additional evidence. Other types of evidence also supported the importance of developing students' ability to understand fractions on a number line. Students' ability to locate decimals on a number line is related to general math achievement. A study of 5th- and 6th-grade German students found a significant positive correlation between students' skill in estimating the location of decimals on a number line and their self-reported mathematics grades in school.[200] In addition, a mathematician's analysis indicated that learning to represent the full range of numbers on number lines is fundamental to understanding numbers.[201]

Recommendation 3. Help students understand why procedures for computations with fractions make sense.

Level of evidence: Moderate Evidence

The panel rated this recommendation as being supported by *moderate evidence*, based on studies specifically related to conceptual and procedural knowledge of fractions. This evidence rating is based on three randomized controlled trials that met WWC standards and demonstrated the effectiveness of teaching conceptual understanding when developing students' computational skill with decimals.[202] Interventions that iterated between instruction on conceptual knowledge and procedural knowledge had a positive effect on decimal computation.[203] Although the studies focused on decimals and were relatively small-scale, the panel believes that the three, together with the extensive evidence that meaningful information is remembered much better than meaningless information,[204] provide persuasive evidence for the recommendation. Additional support for the recommendation comes from four correlational studies of 4th-, 5th- and 6th-grade students that showed significant relations between conceptual and procedural knowledge of fractions.[205] Consensus documents, such as *Adding It Up* and the National Mathematics Advisory Panel report, also suggest the importance of combining instruction on conceptual understanding with procedural fluency.[206]

Panel members focused their review on studies that specifically examined interventions to develop students' understanding of fraction computation. Three randomized controlled trials that met WWC standards support the recommendation.[207] Two of the studies used computer-based interventions to compare different ways of ordering conceptual and procedural instruction for 6th-grade students.[208] The studies' treatment groups alternated between conceptual lessons on decimal place value and procedural lessons on addition and subtraction of decimals; the control groups completed all of the conceptual lessons before receiving any of the procedural lessons. The intervention consisted of six lessons, during which students solved word problems while receiving feedback from the computer program as needed. Both of the relatively small-scale studies found positive effects of iterating between conceptual and procedural lessons. One randomly assigned 26 students and found a large, significant effect on computational proficiency with decimals (effect size = 2.38); the other study randomly assigned four classrooms and found a substantively important, but not significant, effect (effect size = 0.63).

The third study examined an intervention designed to improve students' conceptual understanding of how to locate decimals on a number line.[209] In it, 5th- and 6th-grade students practiced locating fractions on a number line using a computer-based game called Catch the Monster. Students in the treatment groups received either a prompt to notice the tenths digit or a number line divided into tenths—two interventions that the panel views as building students' conceptual knowledge. Control students did not receive the prompts and used a 0-to-1 number line without the tenths marked. Both treatments had a significant, positive effect on students' ability to locate decimals on a number line without the prompts or the tenths marked. Receiving both the prompts and the number line with the tenths marked had a greater impact than receiving the two interventions separately.

The panel's recommendation also is supported by correlational evidence that shows a significant relation between students' conceptual and procedural knowledge of fractions. Hecht et al. (2003) administered a variety of assessments to 105 5th-graders, and Hecht (1998) assessed 103 7th- and 8th-graders to examine how conceptual understanding and procedural skill are related. Hecht and Vagi (in press) included a sample of 181 4th- and 5th-graders to measure the relation between conceptual and procedural knowledge. The studies measured both conceptual and

Table D.2. Studies of interventions that developed conceptual understanding of fraction computation that met WWC standards (with or without reservations)

Citation	Grade Level	Analysis Sample Size	Intervention	Comparison	Outcome	Effect Size[210]
Rittle-Johnson and Koedinger (2002)	6th	4 classrooms	Students complete six computer-based lessons on computation with decimals, alternating between conceptual and procedural lessons.	Students complete six computer-based lessons on computation with decimals, completing all of the conceptual lessons before the procedural lessons.	Computational proficiency with decimals	0.63, ns
Rittle-Johnson and Koedinger (2009)	6th	26 students	Students complete six computer-based lessons on computation with decimals, alternating between conceptual and procedural lessons.	Students complete six computer-based lessons on computation with decimals, completing all of the conceptual lessons before the procedural lessons.	Decimal arithmetic	2.83, sig
Rittle-Johnson, Siegler, and Alibali (2001)	5th and 6th	61 students	When locating decimals on a number line, students receive a prompt to notice the tenths digit and use a 0-to-1 number line with the tenths marked.	When locating decimals on a number line, students used a 0-to-1 number line without the tenths marked.	Locating decimals on a number line	0.57, sig

ns = not significant
sig = statistically significant

procedural knowledge of fractions and fraction computation. All three studies found that after controlling for other factors, conceptual knowledge of fractions significantly predicted students' ability to succeed at fraction computation and estimation. While these studies show a correlation between conceptual and procedural knowledge, they do not establish whether interventions to develop conceptual knowledge improve procedural knowledge.

In another experiment, Rittle-Johnson, Siegler, and Alibali (2001)[211] found that 5th-grade students' understanding of decimals (i.e., relative magnitude and equivalence) was significantly related to their ability to locate decimals on a number line.[212] Controlling for initial procedural knowledge, conceptual knowledge was found to account for 20% of performance variance on a test of procedural knowledge.

Manipulatives and representations. The panel identified evidence that supports the first action step, which recommends using manipulatives and visual representations to teach fraction computation. Two randomized controlled trials, both unpublished dissertations, that met WWC standards found that using manipulatives had a positive effect on fraction computation.[213] Nishida (2008)[214] conducted a relatively small-scale study on the use of fraction circles to teach numerator-denominator relations and other fraction concepts. The study found that having students use fraction circles, rather than observing teachers' use of them, significantly improved students' understanding of fraction concepts relevant to computation (effect size = 0.73). The use of manipulative fraction circles also had a substantively important, but not statistically significant, effect on fraction understanding, compared with the use of pictures of fractions circles.

The second study found that using a variety of manipulatives to supplement a 3rd-grade fractions curriculum improved students' understanding of fractions and fraction computation.[215] The study's unit included lessons on fraction magnitude, equivalence, addition, and subtraction. Teachers in the study used many of the same materials, but teachers in the treatment group also employed various manipulatives and models, including fraction squares, fraction games, fraction strips, pizzas, fraction spinners, cubes, grid cards, paper strips, virtual manipulatives, cutouts, and shapes. The use of these manipulatives had a substantively important, but not statistically significant, effect on a textbook assessment of fraction knowledge and computation (effect size = 0.60).

A randomized controlled trial that potentially meets standards examined the use of manipulatives and real-world contexts for teaching fractions.[216] The study examined a curriculum developed by the Rational Number Project (RNP) that employs a multi-pronged approach incorporating manipulatives, real-world contexts, and estimation and focuses on building students' quantitative sense of fractions. Teachers of 5th- and 6th-grade students were randomly assigned to use either the RNP curriculum or one of two commercial curricula that included minimal use of manipulatives (*Addison-Wesley Mathematics* or *Mathematics Plus*). The RNP curriculum had a significant positive effect on fraction computation and estimation (effect size = 0.27 and 0.65, respectively). However, the study provided insufficient information to assess sample attrition, and amount of use of manipulatives was only one of many differences between the curricula, making it difficult to distinguish which aspects of the intervention led to the positive outcomes.

Real-world contexts and intuitive understanding. Use of real-world concepts also can improve fraction computation proficiency (Step 4). A randomized controlled trial that met WWC standards indicated that using information from students to personalize lessons on fraction division significantly improved their ability to solve fraction division word problems.[217] Students in the treatment condition received instruction via computer-assisted lessons based on contexts suggested by the students; control students were taught using abstract lessons without such contexts. The treatment targeted 5th- and 6th-grade students during a single-lesson unit on fraction division.

A quasi-experimental design study that potentially meets standards evaluated the impact of practicing fraction computation with problems set in everyday contexts.[218] Over the course of three days, students in the treatment group solved contextualized problems involving computation with decimals. Problems included references to soft-drink bottles, monetary exchanges, and measurement. The control group solved similar problems but without any contextual references. Based on the author's calculations, instruction using contextualized problems significantly improved the students' ability to order and compare decimals. The study had a small sample of 16 11- and 12-year-olds from New Zealand; it potentially met standards because insufficient information existed to demonstrate that the treatment and control groups were equivalent at baseline.

Recommendation 4. Develop students' conceptual understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems.

Level of evidence: Minimal Evidence

The panel assigned a rating of *minimal evidence* to this recommendation. Evidence for the overall recommendation comes from consensus documents that emphasize the importance of proportional reasoning for mathematics learning.[219] The panel separately reviewed evidence for the three action steps that comprise this recommendation.

These action steps are supported by case studies demonstrating the variety of strategies students use to solve ratio, rate, and proportion problems; a study of manipulatives that met WWC standards; and two studies that met standards and taught strategies for solving word problems.

Building on early-developing strategies for solving proportionality problems. Evidence for the first action step is based on case studies that examine students' strategies for solving proportionality problems. No studies both met standards and examined the effect of using students' developing strategies to improve their understanding of proportionality. However, the panel believes that the findings of these case studies provide a basis for using a progression of problems that builds on these strategies to develop students' proportional reasoning.

A literature review of early proportional reasoning found that students initially tend to rely on strategies that build up additively from one ratio to another.[220] Students who use this approach may not understand the multiplicative relations between ratios.[221] To illustrate this point, a case study of 21 4th- and 5th-grade students described four developmental levels for solving proportionality problems.[222] One important difference among these levels was whether the developmental levels only involved building up from smaller to larger ratios or whether they also included the knowledge that ratios, like fractions, can be reduced.

Carpenter et al. (1999) and Lamon (1994) suggested that treating ratios as single units is an important developmental step for proportional reasoning. In Cramer, Post, and Currier (1993), 8th-grade students were more likely than 7th-grade students to solve proportionality problems by treating the ratio as a unit and by finding an equivalent fraction. Both studies confirm that students have more difficulty with proportionality problems that involve non-integer relations.[223]

Using visual representations and manipulatives. Visual representations and concrete manipulatives can increase students' proficiency in solving rate, ratio, and proportion problems. In a randomized controlled trial that met standards, Fujimura (2001) evaluated the impact of providing students with concrete manipulatives to solve mixture problems. Japanese students in 4th grade received a manipulative to assist them in solving a proportion problem involving the mixture of two liquids. Students used the manipulative to visually represent the unit rate, or the amount of orange concentrate for each unit of water. Completing a problem using the manipulative improved students' ability to later solve the same type of mixture problems without the manipulative. Students in the treatment group performed significantly better than students with no exposure to mixture problems during the intervention (effect size = 0.74). The treatment had a substantively important, but not statistically significant, effect relative to a control group in which students received a worksheet to calculate the unit rate to solve mixture problems (effect size = 0.34).

An instructional strategy that taught students to use a data table for representing information in a missing value proportion problem had a significant positive effect on the students' ability to solve these problems. In a study that met WWC standards, 7th-graders were taught a problem-solving strategy in which they identified the problem type, represented the problem in a table, determined the multiplicative relation between the known quantities, and then applied that relation to calculate the unknown quantity.[224] Researchers randomly assigned five classrooms to receive instruction in either the above strategy or a substitute approach in which students learned to recognize the problem structure, solve the problem by substituting integers for any complex numbers, and then resolve the problem with the complex numbers. After 10 lessons, students in the treatment group performed better than those in the control group on missing value proportion problems.

In another randomized controlled trial that met standards, Terwel et al. (2009) investigated the effectiveness of instructing 5th-grade students to solve percentage problems by constructing representations collaboratively instead of using teacher-made representations and graphs. This intervention had a substantively important, but not statistically significant, impact on student performance on a researcher-constructed posttest of problem solving with percentages (effect size = 0.41).

Strategies for solving word problems. The literature on teaching strategies for word problems includes many studies outside the scope of this guide—studies that focus on students in 9th grade or above, low-performing students, and students with learning disabilities or on topics other than ratio, rate, or proportion.[225] In its review of available research, the National Mathematics Advisory Panel used these studies to support the teaching of explicit strategies for solving word problems with low-performing students and students with learning disabilities.[226] However, for this action step, the panel sought evidence specifically related to students without diagnosed learning disabilities up to 8th grade and to ratio, rate, and proportion word problems.

Two randomized controlled trials that met standards examined a four-step strategy for teaching students to solve ratio and proportion word problems.[227] The strategy involved a schema-based approach in which students identify the problem type, represent critical information from the problem in a diagram, translate information into a mathematical equation, and solve the problem. Key aspects of the approach, which was designed to address concerns about the limitations of direct instruction, include teaching students to identify underlying problem structures, such as through schematic diagrams, and comparing and contrasting different solution strategies and problem types. One of the studies focused on students with learning disabilities (i.e., 16 of the 19 students had a diagnosed learning disability),[228] and the other included students with a more diverse ability range.[229] Xin, Jitendra, and Deatline-Buckman (2005) found a significant positive effect (albeit with students with learning problems) of an approach that taught students to identify the problem type and represent the problem using a diagram. Students in the comparison group also learned strategies for solving word problems but focused more on drawing pictures to represent the problems. Jitendra et al. (2009) found a substantively important, but not statistically significant, effect of teaching the four-step strategy on researcher-developed tests of ratio and proportion word problems, relative to teaching word problems with a district-adopted mathematics textbook (effect size = 0.33 and 0.38, immediate and delayed posttests, respectively).[230]

A third randomized controlled trial, Moore and Carnine (1989), also examined an explicit strategy for teaching students to solve ratio and proportion word problems. This study met standards but is outside the review protocol because it included students in 9th through 11th grades and focused on special education and low-performing students. The panel views the study as providing supplemental evidence to support the recommendation. The WWC did not have sufficient information to calculate effect sizes, but the study's authors report that teaching students explicit rules and problem-solving strategies significantly improved their proficiency in solving ratio word problems relative to students taught using a basal curriculum.

Recommendation 5. Professional development programs should place a high priority on improving teachers' understanding of fractions and of how to teach them.

Level of evidence: Minimal Evidence

The panel assigned a *minimal evidence* rating to this recommendation because of limited rigorous evidence on the effects of fractions-related professional development activities.

To evaluate this recommendation, the panel sought evidence that professional development that focuses specifically on fractions improves student outcomes.[231] Two studies that focused on developing teachers' knowledge of fractions met standards. The professional development in the first study addressed the first two action steps for the recommendation and found positive effects on student learning; the second study addressed all three action steps but did not find a significant effect on students' understanding of fractions.[232] Two other studies met standards and provided evidence for the recommendation's third step—developing teachers' understanding of students' mathematical thinking—but focused on whole number addition and algebraic reasoning rather than on fractions.[233] A handful of other studies potentially met standards but did not examine fractions or did not provide professional development directly relevant to the recommendation.[234]

Despite the limited evidence on the effects of professional development activities on teachers' understanding of fraction concepts and skills, the panel believes the need to develop teachers' knowledge of fractions and of how to teach them is critical. Teachers' mathematical content knowledge is positively correlated with students' mathematics achievement,[235] and researchers have consistently found that teachers in the United States lack a deep conceptual understanding of fractions.[236] Taken together, these findings suggest that providing professional development on fraction concepts is important.

Professional development related to fractions. One random assignment study met standards and examined a professional development program called Integrated Mathematics Assessment (IMA). This program addressed teachers' understanding of (1) fraction concepts, (2) how students learn fractions, (3) students' motivation for math achievement, and (4) assessment.[237] Teachers learned about fraction concepts through activities and exercises that were more complex versions of those for students. To understand students' thinking, teachers examined student work and videotapes of students solving problems and explored students' difficulties in learning fractions. The IMA training consisted of a five-day summer institute and 13 follow-up sessions for upper elementary teachers. Teachers assigned to the IMA professional development program achieved a significant improvement in their students' conceptual understanding of fractions, compared with teachers in the teacher support group, who met nine times to reflect on their instructional practices. The IMA training had a substantively important, but not statistically significant, effect on students' ability to compute with fractions.

A more recent study of professional development related to fractions also met WWC standards but did not find a significant effect on students' learning of fraction concepts.[238] The study examined two professional development programs for 7th-grade teachers in 12 districts across the country. Teachers in the treatment schools were eligible for about 68 hours of training through a three-day summer institute and five 1-day seminars paired with two-day in-school coaching visits. The professional development focused on conceptual and procedural skill in rational number topics, as well as mathematics knowledge for teaching. This included identifying the key aspects of mathematical understanding, recognizing common errors made by students, and selecting representations for teaching fractions. Activities included solving math problems and receiving feedback on their solutions, discussing common student misconceptions, and planning lessons. Teachers in the control schools received the existing professional development provided by the district. However, the professional development did not have a significant impact on students' understanding of fractions, decimals, percentages, or proportions.

Professional development related to other mathematics topics. Finding little evidence related specifically to fractions, the panel expanded its review to include professional development that focused on other

math topics. Two additional studies met standards and implemented training relevant to the third action step of the recommendation—developing teachers' understanding of students' mathematical thinking.[239]

One study examined a four-week summer workshop (20 hours) aimed at developing teachers' knowledge of how children learn whole number addition and subtraction concepts.[240] Teachers participating in the program, called Cognitively Guided Instruction (CGI), learned about children's solution strategies and how to classify problem types, discussed how to incorporate information from the CGI workshop into the classroom, and planned instruction accordingly. Compared with a control group of teachers who received four hours of workshops on problem solving and the use of nonroutine problems, the CGI program had a substantively important, but not statistically significant, effect on computation problems and addition and subtraction word problems. The differing amounts of time that teachers spent in the two conditions also limited interpretation of the findings.

The second study, a randomized controlled trial study conducted in a large, urban district with 1st- through 5th-grade teachers, also examined a professional development program focused on developing teachers' understanding of students' mathematical thinking.[241] Emphasizing understanding of the equal sign and using number relations to simplify calculations, the training in this study was designed to improve teachers' ability to incorporate algebraic reasoning into elementary mathematics. Teachers learned to make sense of students' strategies for solving problems, to link students' thinking to key mathematical ideas, and to lead mathematical conversations with students. The program included an initial meeting and eight monthly after-school work-group meetings (a total of about 16.5 hours), as well as a trainer who spent a half-day a week at each school to provide additional support. Results from the study showed that this professional development significantly improved students' understanding of the equal sign and students' use of relational-thinking strategies for solving computations but not for solving equations (i.e., with letters representing unknown quantities).

Additional evidence. There is further evidence that students' achievement is positively related to teachers' mathematics knowledge for teaching—for example, their skill at explaining math concepts, understanding student strategies, and providing representations. A study of 699 1st- and 3rd-grade math teachers found a positive relation between teachers' math knowledge for teaching and students' learning gains in math after controlling for student and teacher characteristics.[242] Although this study did not specifically focus on fractions, it demonstrated the importance of teachers' math content knowledge for teaching.

Professional development with fractions is needed because many U.S. teachers lack a deep conceptual understanding of fractions.[243] A study comparing Chinese and American teachers found that only 9 of the 21 U.S. teachers who tried to calculate $1\frac{3}{4} \div \frac{1}{2}$ did so correctly, whereas all 72 Chinese teachers correctly completed the problem.[244] U.S. teachers could not represent or explain division with fractions, and many confused the algorithm for dividing fractions with the algorithms for adding, subtracting, and multiplying fractions.

Other studies have reported similar findings. A study of 218 elementary school teachers in Minnesota and Illinois found that many teachers could not solve computation problems involving fractions and that most of those who correctly solved problems could not provide a correct explanation of their solutions.[245] For example, almost half of the teachers in the Minnesota study incorrectly solved a subtraction problem involving fractions ($\frac{1}{3} - \frac{3}{7}$). Further, a study of 46 preservice middle school teachers at a university in Texas found that most teachers knew the procedure for dividing with fractions but did

not understand why the procedure worked and could not judge whether an alternative procedure for solving a division problem with fractions was correct.[246]

These studies clearly indicate that teachers' understanding of fractions needs to be upgraded. However, the recommendation regarding professional development is largely based on the panel's expertise, because of the limited evidence regarding the effects of professional development activities focused on fractions.

Appendix E

Evidence Heuristic

This appendix contains a heuristic for categorizing the evidence base for practice guide recommendations as *strong evidence*, *moderate evidence*, or *minimal evidence*. This heuristic is intended to serve as a framework to ensure that the levels of evidence are consistently applied across practice guides while at the same time clarifying the levels for panelists and educators. The core document to accompany this heuristic is the "Institute of Education Sciences levels of evidence for practice guides" (Table 1 in this practice guide).

Table E.1. Evidence heuristic

Criteria for a Strong Evidence Base	This criterion is necessary for a strong level of evidence.
High internal validity (high-quality causal designs). Studies must meet WWC standards with or without reservations.[247]	✓
High external validity (requires a quantity of studies with high-quality causal designs). Studies must meet WWC standards with or without reservations.[248]	✓
Effects on relevant outcomes—consistent positive effects without contradictory evidence (i.e., no statistically significant negative effects) in studies with high internal validity.[249]	✓
Direct relevance to scope (i.e., ecological validity)—relevant context (e.g., classroom vs. laboratory), sample (e.g., age and characteristics), and outcomes evaluated.	✓
Direct test of the recommendation in the studies, or the recommendation is a major component of the intervention tested in the studies.	✓
For assessments, meets *The Standards for Educational and Psychological Testing.*	✓
Panel has a high degree of confidence that this practice is effective.	✓
Criteria for a Moderate Evidence Base	**This criterion is necessary for a moderate level of evidence.**
High internal validity but moderate external validity (i.e., studies that support strong causal conclusions, but generalization is uncertain) OR studies with high external validity but moderate internal validity (i.e., studies that support the generality of a relation, but the causality is uncertain). ▪ The research may include studies generally meeting WWC standards and supporting the effectiveness of a program, practice, or approach with small sample sizes and/or other conditions of implementation or analysis that limit generalizability. ▪ The research may include studies that support the generality of a relation but do not meet WWC standards; [250] however, they have no major flaws related to internal validity other than lack of demonstrated equivalence at pretest for quasi-experimental design studies (QEDs). QEDs without equivalence must include a pretest covariate as a statistical control for selection bias. These studies must be accompanied by at least one relevant study meeting WWC standards.	✓
Effects on relevant outcomes—a preponderance of evidence of positive effects. Contradictory evidence (i.e., statistically significant negative effects) must be discussed by the panel and considered with regard to relevance to the scope of the guide and intensity of the recommendation as a component of the intervention evaluated.	✓
Relevance to scope (i.e., ecological validity) may vary, including relevant context (e.g., classroom vs. laboratory), sample (e.g., age and characteristics), and outcomes evaluated.	
Intensity of the recommendation as a component of the interventions evaluated in the studies may vary.	
For assessments, evidence of reliability that meets *The Standards for Educational and Psychological Testing* but with evidence of validity from samples not adequately representative of the population on which the recommendation is focused.	✓
The panel is not conclusive about whether the research has effectively controlled for other explanations or whether the practice would be effective in most or all contexts.	
The panel determines that the research does not rise to the level of strong evidence but is more compelling than a minimal level of evidence.	✓

(continued)

Table E.1. Evidence heuristic *(continued)*

Criteria for a Minimal Evidence Base	This criterion is necessary for a minimal level of evidence.
Expert opinion based on defensible interpretations of theory (or theories). In some cases, this simply means that the recommended practices would be difficult to study in a rigorous, experimental fashion; in other cases, it means that researchers have not yet studied this practice.	
Expert opinion based on reasonable extrapolations from research: ▪ The research may include evidence from studies that do not meet the criteria for moderate or strong evidence (e.g., case studies, qualitative research). ▪ The research may be out of the scope of the practice guide. ▪ The research may include studies for which the intensity of the recommendation as a component of the interventions evaluated in the studies is low.	
There may be weak or contradictory evidence.	
In the panel's opinion, the recommendation must be addressed as part of the practice guide; however, the panel cannot point to a body of research that rises to the level of moderate or strong.	✓

Endnotes[a]

1. For more information, see the WWC Frequently Asked Questions page for practice guides, http://ies.ed.gov/ncee/wwc/references/idocviewer/doc.aspx?docid=15.
2. See the WWC guidelines at http://ies.ed.gov/ncee/wwc/pdf/wwc_procedures_v2_standards_handbook.pdf.
3. This includes randomized control trials (RCTs), quasi-experimental designs (QEDs), regression discontinuity designs (RDDs), and single-case designs (SCDs) evaluated with WWC standards.
4. If the only evidence meeting standards (with or without reservations) is SCDs, the guidelines set by the SCD standards panel will apply. For external validity, the requirements are a minimum of five SCD research papers examining the intervention that meet evidence standards or meet evidence standards with reservations, the studies must be conducted by at least three different research teams at three different geographical locations, and the combined number of experiments across studies totals at least 20.
5. In certain circumstances (e.g., a comparison group cannot be formed), the panel may base a moderate rating on multiple correlational designs with strong statistical controls for selection bias that demonstrate consistent positive effects without contradictory evidence.
6. Baldi et al. (2007); Gonzales et al. (2009).
7. National Academy of Sciences (2007).
8. McCloskey (2007); National Academy of Sciences (2007); Rivera-Batiz (1992).
9. Starkey, Klein, and Wakeley (2004).
10. Rivera-Batiz (1992).
11. Mullis et al. (1997).
12. National Mathematics Advisory Panel (2008).
13. Hoffer et al. (2007).
14. National Council of Teachers of Mathematics (2007).
15. Kloosterman (2010).
16. **Rittle-Johnson, Siegler, and Alibali (2001)**.
17. Vamvakoussi and Vosniadou (2004).
18. Davis and Pepper (1992); Frydman and Bryant (1988); Hunting and Sharpley (1988); Pepper (1991); Singer-Freeman and Goswami (2001); Spinillo and Bryant (1991, 1999).
19. Ma (1999).
20. Ma (1999); Moseley, Okamoto, and Ishida (2007).
21. Davis and Pepper (1992); Frydman and Bryant (1988); Hunting and Sharpley (1988); Pepper (1991).
22. Singer-Freeman and Goswami (2001); Spinillo and Bryant (1991, 1999).
23. Davis and Pepper (1992); Frydman and Bryant (1988); Hunting and Sharpley (1988); Pepper (1991); Singer-Freeman and Goswami (2001); Spinillo and Bryant (1991, 1999).
24. Empson (1999); Streefland (1991).
25. Davis and Pepper (1992); Frydman and Bryant (1988); Hunting and Sharpley (1988); Pepper (1991).
26. Chen (1999); Hunting and Sharpley (1988).
27. Empson (1999); Streefland (1991).
28. Singer-Freeman and Goswami (2001); Spinillo and Bryant (1991, 1999).
29. Spinillo and Bryant (1991, 1999).
30. Goswami (1989, 1995).
31. Empson (1999); Streefland (1991).
32. Empson (1999).
33. Teachers should also note that sharing situations with more people than objects result in proper fractions, whereas sharing situations with more objects than people result in improper fractions or mixed numbers.
34. Empson (1999); Streefland (1991).
35. Empson (1999).
36. Pothier and Sawada (1990).
37. Ibid.
38. Empson (1999); Streefland (1991).
39. Ibid.
40. Streefland (1991).
41. Ibid.
42. Resnick and Singer (1993).
43. Ibid.
44. Warren and Cooper (2007).
45. Ibid.

[a] Eligible studies that meet WWC evidence standards or meet evidence standards with reservations are indicated by **bold text** in the endnotes and references pages. For more information about these studies, please see Appendix D.

46. Davis and Pitkethly (1990).
47. Frydman and Bryant (1988).
48. Ibid.
49. Mack (1993).
50. Sophian (2007).
51. **Booth and Siegler (2008); Ramani and Siegler (2008); Siegler and Booth (2004); Siegler and Ramani (2008, 2009)**.
52. **Rittle-Johnson, Siegler, and Alibali (2001)**.
53. Booth and Siegler (2006, **2008**); Schneider, Grabner, and Paetsch (2009); **Siegler & Booth (2004)**.
54. **Ramani and Siegler (2008); Siegler and Ramani (2008, 2009)**.
55. **Siegler and Booth (2004)**.
56. **Rittle-Johnson, Siegler, and Alibali (2001)**.
57. Keijzer and Terwel (2003).
58. Ibid.
59. Bright et al. (1988); Izsak, Tillema, and Tunc-Pekkan (2008).
60. Bright et al. (1988).
61. Schneider, Grabner, and Paetsch (2009).
62. Wu (2002).
63. Sophian (2007).
64. Izsak, Tillema, and Tunc-Pekkan (2008); Yanik, Helding, and Flores (2008).
65. Adapted from Beckmann (2008); Wu (2002).
66. Beckmann (2008); Lamon (2005).
67. Van de Walle, Karp, and Bay-Williams (2010).
68. Bright et al. (1988); Izsak (2008).
69. Ni and Zhou (2005).
70. Lamon (2005); Niemi (1996).
71. Ni and Zhou (2005).
72. Bright et al. (1988).
73. **Rittle-Johnson and Koedinger (2002, 2009); Rittle-Johnson, Siegler, and Alibali (2001)**.
74. Anderson (2004); Schneider and Pressley (1997).
75. Hecht (1998); Hecht, Close, and Santisi (2003); Hecht and Vagi (in press); **Rittle-Johnson, Siegler, and Alibali (2001)**.
76. **Rittle-Johnson and Koedinger (2002, 2009)**.
77. **Rittle-Johnson, Siegler, and Alibali (2001), Experiment 2**.
78. Hecht (1998); Hecht, Close, and Santisi (2003); Hecht and Vagi (in press).
79. Rittle-Johnson, Siegler, and Alibali (2001), Experiment 1.
80. **Hudson Hawkins (2008); Nishida (2008)**.
81. **Nishida (2008)**, **Experiment 2.**
82. **Hudson Hawkins (2008)**.
83. Cramer, Post, and delMas (2002).
84. **Anand and Ross (1987)**; Irwin (2001).
85. **Anand and Ross (1987)**.
86. Irwin (2001).
87. Mack (2000); Mack (2001); Rittle-Johnson and Koedinger (2005).
88. Cramer and Wyberg (2009).
89. Sowder et al. (1998).
90. Bulgar (2009).
91. Van de Walle, Karp, and Bay-Williams (2010).
92. Smith (2002).
93. Hecht (1998).
94. Johanning (2006).
95. Smith (2002).
96. Gerver and Sgroi (1989).
97. Cramer and Wyberg (2009); Cramer, Wyberg, and Leavitt (2008); Hecht (2003).
98. Kerslake (1986); Mack (1995); Painter (1989).
99. Sharp (2004).
100. Tatsuoka and Tatsuoka (1983).
101. Hackenberg (2007).
102. Mack (1995).
103. Painter (1989).
104. Ashlock (2009); Barash and Klein (1996).
105. Kamii and Warrington (1995); Mack (2001); Warrington and Kamii (1998).
106. Mack (1990, 1993).
107. **Anand and Ross (1987)**; Irwin (2001).
108. Mack (1990).
109. Ibid.
110. Cramer, Wyberg, and Leavitt (2008).

111. Lesh, Post, and Behr (1988); National Council of Teachers of Mathematics (2000); National Mathematics Advisory Panel (2008).
112. Smith (2002); Stemn (2008).
113. National Mathematics Advisory Panel (2008).
114. Kloosterman (2010); Lamon (2007); Thompson and Saldanha (2003).
115. Lesh, Post, and Behr (1988); National Council of Teachers of Mathematics (2000); National Mathematics Advisory Panel (2008).
116. Carpenter et al. (1999); Cramer, Post, and delMas (2002); Lamon (1994).
117. **Fujimura (2001)**.
118. **Sellke, Behr, and Voelker (1991)**.
119. **Terwel et al. (2009)**.
120. **Jitendra et al. (2009); Xin, Jitendra, and Deatline-Buchman (2005)**.
121. The study with mostly learning disabled students falls within the panel's protocol because some of the students are low performing but not learning disabled. The protocol includes studies that did not focus solely on learning disabled students.
122. This step focuses primarily on strategies for solving two types of proportion problems (Lamon, 2005): (1) missing-value problems in which students are given one complete ratio and another ratio with a missing value that students must identify, and (2) comparison problems in which students determine whether two ratios are equivalent.
123. Carpenter et al. (1999); Cramer, Post, and Currier (1993); Lesh, Post, and Behr (1988).
124. Carpenter et al. (1999).
125. Ibid.
126. Carpenter et al. (1999); Lamon (2007).
127. Kent, Arnosky, and McMonagle (2002); Terwel et al. (2009).
128. Resnick and Singer (1993).
129. Lamon (1993).
130. **Jitendra et al. (2009)**.
131. **Xin, Jitendra, and Deatline-Buchman (2005)**.
132. Hembree (1992); **Jitendra et al. (2009)**.
133. **Jitendra et al. (2009); Xin, Jitendra, and Deatline-Buchman (2005)**.
134. **Bottge (1999)**; Heller et al. (1989).
135. Li and Kulm (2008); Ma (1999); Newton (2008); Post et al. (1988).
136. Hill, Rowan, and Ball (2005).
137. Li and Kulm (2008); Ma (1999); Newton (2008); Post et al. (1988).
138. **Saxe, Gearhart, and Nasir (2001)**.
139. **Garet et al. (2010)**.
140. **Garet et al. (2010)**; Ma (1999); Post et al. (1988).
141. Ma (1999).
142. Post et al. (1988).
143. Hill et al. (2005).
144. Ma (1999).
145. Li and Kulm (2008).
146. Although a full discussion of how to structure professional development is beyond this guide's scope, the panel provides basic suggestions that address the goal of this step.
147. Marchiondra (2006); **Saxe, Gearhart, and Nasur (2001)**.
148. Tirosh (2000).
149. **Saxe, Gearhart, and Nasir (2001)**; Wu (2004).
150. **Saxe, Gearhart, and Nasir (2001)**.
151. **Carpenter et al. (1989)**.
152. **Jacobs et al. (2007); Saxe, Gearhart, and Nasir (2001)**.
153. Kloosterman (2010); National Council of Teachers of Mathematics (2007).
154. Mullis et al. (1997).
155. National Research Council (2001).
156. Sophian and Wood (2007).
157. Cramer and Post (1993); Van de Walle, Karp, and Bay-Williams (2010).
158. Davis and Pepper (1992); Frydman and Bryant (1988); Hunting and Sharpley (1988); Pepper (1991); Singer-Freeman and Goswami (2001); Spinillo and Bryant (1991, 1999).
159. Empson (1999); Streefland (1991).
160. Chen (1999); Davis and Pepper (1992); Frydman and Bryant (1988); Hunting and Sharpley (1988); Pepper (1991).
161. Chen (1999).
162. Hunting and Sharpley (1988).
163. Sophian, Garyantes, and Chang (1997).
164. Ibid.

165. Empson (1999); Streefland (1991).
166. Streefland (1991).
167. Empson (1999).
168. Singer-Freeman and Goswami (2001); Spinillo and Bryant (1991, 1999).
169. Singer-Freeman and Goswami (2001).
170. Spinillo and Bryant (1991).
171. Spinillo and Bryant (1991, 1999).
172. Boyer, Levine, and Huttenlocher (2008); Singer-Freeman and Goswami (2001); Spinillo and Bryant (1999).
173. Goswami (1989).
174. Goswami (1995).
175. Spinillo (1995).
176. **Ramani and Siegler (2008); Siegler and Ramani (2008, 2009)**.
177. **Rittle-Johnson, Siegler, and Alibali (2001)**.
178. Booth and Siegler (2006, 2008); Siegler and Booth (2004); **Siegler and Ramani (2009)**.
179. Opfer and DeVries (2008).
180. **Ramani and Siegler (2008); Siegler and Ramani (2008, 2009)**.
181. **Ramani and Siegler (2008); Siegler and Ramani (2008)**.
182. **Siegler and Ramani (2009)**.
183. **Ramani and Siegler (2008); Siegler and Ramani (2008, 2009)**. The WWC calculated the effect size for **Siegler and Ramani (2008)**; the effect sizes reported for **Ramani and Siegler (2008)** and **Siegler and Ramani (2009)** were reported by the authors (the authors did not provide sufficient information for the WWC to calculate the effect size).
184. **Siegler and Ramani (2008); Ramani and Siegler (2008). Siegler and Ramani (2009)** found a significant effect of the number board game on percent absolute error but did not report an effect size or provide sufficient information to calculate it.
185. **Siegler and Ramani (2008, 2009)**.
186. **Siegler and Booth (2004)**.
187. The authors did not report whether the difference between treatment and control groups was significant.
188. Following WWC guidelines, improved outcomes are indicated by either a positive statistically significant effect or a positive substantively important effect size. In this guide, the panel discusses substantively important findings as ones that contribute to the evidence of practices' effectiveness, even when those effects are not statistically significant. See the WWC guidelines at http://ies.ed.gov/ncee/wwc/pdf/wwc_procedures_v2_standards_handbook.pdf.
189. **Booth and Siegler (2008)**.
190. This study also included a circular numeric board game treatment that was not considered as part of the evidence for this recommendation.
191. This study also included kindergarten students, but the authors focused their analysis results on 1st- and 2nd-graders since the treatment did not have the expected effect for kindergarteners.
192. This study also included a treatment group that had students generate a number line with the addends and sums (instead of providing them with a computer-generated line). These two treatments did not significantly differ in the percentage of correct answers.
193. Booth and Siegler (2006); Siegler and Booth (2004).
194. **Siegler and Booth (2004)**.
195. Booth and Siegler (2006).
196. **Rittle-Johnson, Siegler, and Alibali (2001)**.
197. Keijzer and Terwel (2003).
198. Bright et al. (1988); Izsak, Tillema, and Tunc-Pekkan (2008).
199. Bright et al. (1988).
200. Schneider, Grabner, and Paetsch (2009).
201. Wu (2002).
202. **Rittle-Johnson and Koedinger (2002, 2009); Rittle-Johnson, Siegler, and Alibali (2001)**.
203. **Rittle-Johnson and Koedinger (2002, 2009)**.
204. Anderson (2004); Schneider and Pressley (1997).
205. Hecht (1998); Hecht, Close, and Santisi (2003); Hecht and Vagi (in press); **Rittle-Johnson, Siegler, and Alibali (2001)**.
206. National Mathematics Advisory Panel (2008); National Research Council (2001).

207. **Rittle-Johnson and Koedinger (2002, 2009); Rittle-Johnson, Siegler, and Alibali (2001)**.
208. **Rittle-Johnson and Koedinger (2002, 2009)**.
209. **Rittle-Johnson, Siegler, and Alibali (2001), Experiment 2.**
210. For a p-value < 0.05, the effect size is significant (sig); for a p-value ≥ 0.05, the effect size is not significant (ns).
211. **Rittle-Johnson, Siegler, and Alibali (2001)**, Experiment 1.
212. **Ibid**.
213. **Hudson Hawkins (2008); Nishida (2008)**.
214. **Nishida (2008), Experiment 2.**
215. **Hudson Hawkins (2008)**.
216. Cramer, Post, and delMas (2002).
217. **Anand and Ross (1987)**.
218. Irwin (2001).
219. National Council of Teachers of Mathematics (2000); National Mathematics Advisory Panel (2008).
220. Resnick and Singer (1993).
221. Lamon (1994).
222. Carpenter et al. (1999).
223. Carpenter et al. (1999); Cramer, Post, and Currier (1993).
224. **Sellke, Behr, and Voelker (1991)**.
225. Bassok (1990); Carroll (1994); Cooper and Sweller (1987); Lewis (1989); Lewis and Mayer (1987); Reed and Bolstad (1991); Sweller and Cooper (1985).
226. National Math Advisory Panel (2008).
227. **Jitendra et al. (2009); Xin, Jitendra, and Deatline-Buchman (2005)**.
228. **Xin, Jitendra, and Deatline-Buchman (2005)**.
229. **Jitendra et al. (2009)**.
230. **Jitendra et al. (2009)** reported a significant positive effect of the treatment on the problem solving posttest. However, when the WWC applied a clustering correction, since students in the study were clustered in classrooms, the results were not significant. For an explanation, see the WWC Tutorial on Mismatch. For the formulas the WWC used to calculate the statistical significance, see the WWC Procedures and Standards Handbook.
231. The panel did not review studies that measured the effect of professional development on teacher knowledge, although a review by the National Math Advisory Panel did not identify any studies with a comparison group design.
232. **Garet et al. (2010); Saxe, Gearhart, and Nasir (2001).**
233. **Carpenter et al. (1989); Jacobs et al. (2007)**.
234. Cole (1992); Meyer and Sutton (2006); Niess (2005); Ross, Hogaboam-Gray, and Bruce (2006); Sloan (1993).
235. Hill, Rowan, and Ball (2005).
236. Li and Kulm (2008); Ma (1999); Newton (2008); Post et al. (1988).
237. **Saxe, Gearhart, and Nasir (2001)**. Although the study also used a quasi-experimental design to compare two math curricula, only the professional development portion of this study, which used a random assignment design, is relevant for Recommendation 5.
238. **Garet et al. (2010)**.
239. **Carpenter et al. (1989); Jacobs et al. (2007)**.
240. **Carpenter et al. (1989)**.
241. **Jacobs et al. (2007)**.
242. Hill, Rowan, and Ball (2005).
243. Li and Kulm (2008); Ma (1999); Post et al. (1988).
244. Ma (1999).
245. Post et al. (1988).
246. Li and Kulm (2008).
247. This includes randomized control trials (RCTs), quasi-experimental designs (QEDs), regression discontinuity designs (RDDs), and single-case designs (SCDs) evaluated with WWC standards.
248. If the only evidence meeting standards (with or without reservations) is SCDs, the guidelines set by the SCD standards panel will apply. For external validity, the requirements are a minimum of five SCD research papers examining the intervention that meet evidence standards or meet evidence standards with reservations, the studies must be conducted by

at least three different research teams at three different geographical locations, and the combined number of experiments across studies totals at least 20.

249. When evaluating whether effects are consistent or contradictory, consider the psychometric properties of the assessments. For example, effects are less likely to be detected if an assessment is unreliable. Psychometric properties to consider include reliability, the presence of limited or constrained variance, the assessment's ceiling and floor, the assessment's item gradients, whether the assessment was overaligned with the intervention, and the appropriateness of the assessment for the sample to which it was applied.

250. In certain circumstances (e.g., a comparison group cannot be formed), the panel may base a moderate rating on multiple correlational designs with strong statistical controls for selection bias that demonstrate consistent positive effects without contradictory evidence.

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[a]Eligible studies that meet WWC evidence standards or meet evidence standards with reservations are indicated by **bold text** in the endnotes and references pages. For more information about these studies, please see Appendix D.

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