Synthesis of IES-Funded Research on Mathematics: 2002–2013

Authors
Bethany Rittle-Johnson, *Vanderbilt University*
Nancy C. Jordan, *University of Delaware*

National Center for Education Research (NCER)
Meredith Larson (Project Officer)
Christina Chhin

National Center for Special Education Research (NCSER)
Robert Ochsendorf*, May 2014 –March 2016*

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Disclaimer

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Ruth Neild, *Deputy Director for Policy and Research, Delegated Duties of the Director*

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Thomas W. Brock, *Commissioner*

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Disclosure of Potential Conflicts of Interest

The authors of research syntheses are individuals who are nationally recognized experts on the topics that they are synthesizing. IES expects that such experts will be involved professionally in a variety of matters that relate to their work as authors. Prior to engagement in the task, authors are asked to disclose their professional involvements, and institute deliberative processes that encourage critical examination in relation to the content of the research synthesis. Prior to publication, the research synthesis is analyzed by independent external peer review with particular focus on whether the evidence related to the conclusions in the research synthesis has been appropriately presented.

The professional engagements reported by each author that appear most closely associated with the synthesis findings are noted below.

Dr. Nancy C. Jordan is Professor in the School of Education at the University of Delaware. Her research expertise is in mathematical learning, especially children with learning difficulties and disabilities. Dr. Jordan is currently developing a number sense screener for children from Pre-K through first grade and a fraction intervention for struggling learners in middle school. She has served as Principal Investigator for IES grants R324C1000004, R305A150545, and R305B130012. Findings from one of these grants are reported in this research synthesis. No conflicts of interest are present.

Dr. Bethany Rittle-Johnson is Professor in Psychology and Human Development, Peabody College at Vanderbilt University. Her research expertise is in how children learn math. Dr. Rittle-Johnson is currently exploring the relations between early math knowledge, such as pattern knowledge, and later mathematics development. She has served as co-Principal Investigator or Investigator for IES grants R305H050179, R305B080008 and R305A140126. Findings from one of these grants are reported in this research synthesis. No conflicts of interest are present.

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Executive Summary

Proficiency in mathematics is critical to academic, economic, and life success. Greater mathematics knowledge is related to college completion, higher earnings, and better health decisions (Adelman 2006). However, international and national assessment data indicate that U.S. mathematics education is not as effective as it needs to be (Programme for International Student Assessment [PISA]),[1] Trends in International Mathematics and Science Study (TIMSS),[2] and National Assessment of Educational Progress (NAEP).[3] To address this national need to improve mathematics education, the Institute of Education Sciences (IES) funded almost 200 grants on mathematics learning and teaching between 2002 and 2013 through its National Center for Education Research (NCER) and National Center for Special Education Research (NCSER).[4] These projects have focused on developing and evaluating instructional strategies and materials for improving mathematics learning, and approaches to teacher professional development. For this document, the authors synthesized peer-reviewed publications that were products of IES-funded research projects that focused on mathematics teaching and learning for students in kindergarten through high school. This executive summary highlights the primary findings.

Improving Mathematics Learning

IES has funded numerous grants to develop and test instructional strategies and instructional materials for improving students' mathematics learning. This research has focused on a variety of mathematics topics, ranging from basic numeracy to algebra, and includes research on students in general education settings as well as students with or at risk for learning difficulties and disabilities. Results of IES-funded research for improving mathematics learning are organized into two sections based on mathematics topics and grade level: (1) Whole Numbers, Operations, and Word Problem Solving in Elementary School, and (2) Fractions and Algebra in Middle School.

Whole Numbers, Operations, and Word Problem Solving in Elementary School

IES-funded research has made the following contributions.

Contribution 1. Measures emphasizing number, number relations, and number operations reliably identify students who are at risk for mathematics difficulties or disabilities.

Contribution 2. Early number competencies are malleable and can be taught successfully to students with and without mathematics difficulties through targeted and conceptually driven instruction.

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[1] http://www.oecd.org/pisa/keyfindings/pisa-2012-results.htm
[2] http://nces.ed.gov/TIMSS/results11_math11.asp
[3] http://nces.ed.gov/programs/coe/indicator_cnc.asp

[4] NCSER first began funding research in 2006.

Contribution 3. Incorporating activities with number lines reveals and supports students' knowledge of whole numbers.

Contribution 4. Working memory capacity and computational fluency predict word problem-solving accuracy in the early grades.

Contribution 5. Training in how to use learning strategies improves word problem-solving skills in at-risk learners, although response to training may vary according to a student's working memory capacity.

Contribution 6. Dynamic assessments involving teacher-student interaction may improve assessment accuracy and instruction for students at risk for mathematics difficulties.

Contribution 7. In contrast to popular belief, building blocks, "play money," and other manipulatives sometimes have limited value in teaching elementary school mathematics. Manipulatives and materials that have minimal visual distractions can be more effective than ones that are more realistic or complex.

Contribution 8. Improving students' general reasoning skills may also improve their ability to learn mathematics.

Contribution 9. Simple changes in the formatting of arithmetic problems can help improve students' understanding of the equal sign.

Contribution 10. A supplemental mathematics curriculum that integrates the knowledge, skills, and teaching approaches used by Alaska Native people improves Alaskan students' mathematics knowledge.

Fractions and Algebra in the Middle Grades

IES-funded research has made the following contributions:

Contribution 11. A constellation of processes influences fraction learning, including numerical magnitude understanding, arithmetic fluency, attention, memory, and verbal skills.

Contribution 12. Students with and without mathematics difficulties make greater gains when mathematics instruction emphasizes fractions as magnitudes that can be represented on a number line.

Contribution 13. Adolescents with mathematics difficulties benefit from fractions instruction that builds fractions skills and concepts alongside problems anchored in everyday contexts.

Contribution 14. Practice problems should be interleaved so that problems of different types are mixed together rather than grouped together by problem type.

Contribution 15. Comparing multiple ways to solve problems improves student learning, and teachers can help students make effective comparisons.

Contribution 16. Critiquing common incorrect solutions improves student learning.

Contribution 17. Promoting fluency in mapping between different representations of mathematical ideas such as matching a word problem to an appropriate number sentence improves students' learning.

Contribution 18. Producing physical movements and gestures may improve students' mathematical learning.

Contribution 19. Teaching students cognitive strategies for solving word problems, such as categorizing, supports their word problem-solving success more than typical classroom instruction.

Contribution 20. Using computer-based interim assessments provides teachers with diagnostic information and can improve students' mathematics achievement.

Contribution 21. Computer-based tutoring systems can allow for individualized mathematics instruction and have the potential to help students learn mathematics. Systems ranging from a year-long integrated algebra I curriculum to an online assignment system improve students' mathematics learning, although some tutoring systems for raising state test scores among low-income students have shown less promise.

Contribution 22. Using technology to support student collaboration may improve students' learning.

Contribution 23. Improved synthetic speech shows potential to make algebra more accessible for students with visual impairments.

Contribution 24. Increasing instructional time in algebra I for low-performing ninth-grade students can improve their course grades and test scores, but mandating algebra I by ninth grade may not improve test scores or college attendance.

Development and Evaluation of Teacher Professional Development Approaches

Improving student achievement requires changes in teaching practices. Most schools require teachers to attend professional development, and there are countless mathematics teacher professional development programs on the market. However, in a review of teacher professional development programs targeted at improving K–12 mathematics instruction, only two programs were identified that had rigorous evidence for improving student mathematics outcomes (Gersten et al. 2014). In response to the need for evidence-based teacher professional development programs, IES has funded the development and evaluation of several teacher professional development programs focused on mathematics teaching and learning.

IES-funded research has made the following contributions:

Contribution 25. Teacher professional development that helps elementary school teachers build on a previous Pre-K mathematics intervention can boost the long-term effectiveness of the Pre-K intervention.

Contribution 26. Teacher professional development using a Lesson Study approach and targeting specific mathematics content can improve teachers' knowledge and lead to improvements in students' learning of fractions.

Contribution 27. Teacher professional development that helps teachers create a supportive and safe environment for learning can improve the quality of third-grade mathematics instruction, although it does not directly improve end-of-year state test scores.

Contribution 28. Insufficient support from principals and insufficient teacher knowledge are potential barriers to effective teacher professional development.

Summary

The first 11 years of IES-funded research has greatly extended our knowledge about how students learn mathematics and how to improve mathematics outcomes for all students. The research informs decisionmaking in schools in ways that were not considered before the inception of IES. With IES funding, researchers have developed and evaluated effective instructional strategies and materials for teaching all students and specialized interventions for those who are struggling. It has also supported development and evaluation of teacher professional development that is necessary for teachers to improve their integration of more effective instructional strategies and materials. Innovative efforts to translate findings from the education sciences into effective classroom practice must continue through rigorous research in the schools and other educational environments.

Synthesis Context: Why Research on Mathematics Learning Is Needed

Proficiency in mathematics promotes academic, economic, and life success. Academically, the level of mathematics a student completes is a strong predictor of entering college and earning a bachelor's degree (Adelman 2006). Economically, annual income is 65 percent higher among adults who have taken calculus in high school than among adults who have completed only basic mathematics (Altonji, Blom, and Meghir 2012; Rose and Betts 2001). Further, mathematics knowledge at age 7 is a stronger predictor of socioeconomic status (SES) in adulthood than is childhood SES, over and above the effects of IQ, reading achievement, and intelligence (Ritchie and Bates 2013). In part, these trends arise because many careers in promising, well-paying jobs require advanced mathematics for job success. Outside of economic success, mathematics knowledge affects the quality of daily life. For example, many people have insufficient mathematics knowledge to make appropriate health decisions, such as taking appropriate drug dosages and understanding risks and benefits of screenings and treatment (Reyna et al. 2009).

National and international assessments point to the need to improve mathematics education in the United States. In 2015, the National Assessment of Educational Progress (NAEP)—known as "the nation's report card"—found that just 40 percent of fourth-graders and 33 percent of eighth-graders performed at or above proficiency.[5] The goal is for all students to reach proficiency, which is "demonstrated competency over challenging subject matter." On the 2012 Programme for International Student Assessment (PISA), 15-year-old students in the United States performed below average in mathematics relative to the majority of developed countries, with no significant change in average performance over the past 12 years.[6] About a quarter of U.S. students did not reach proficiency in mathematics, with students from lower socioeconomic backgrounds being particularly likely to perform poorly. Similarly, on the 2011 Trends in International Mathematics and Science Study (TIMSS), fourth-grade and eighth-grade students from the U.S. performed significantly worse than students from several other countries, particularly Asian countries.[7] The gap is particularly pronounced among the highest achieving students; over 45 percent of eighth-grade students in Singapore and Korea performed at the advanced level, compared to only 7 percent of students in the United States. Data from the Programme for the International Assessment of Adult Competencies (PIAAC) show a similar stark picture, with U.S. adults ages 16 to 65 seriously underperforming relative to their peers in other Organisation for Economic Co-operation and Development (OECD) countries (Rampey et al. 2016). Mathematics knowledge begins to develop at a young age, and

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[5] http://nces.ed.gov/programs/coe/indicator_cnc.asp
[6] http://www.oecd.org/pisa/keyfindings/pisa-2012-results.htm
[7] http://nces.ed.gov/TIMSS/results11_math11.asp

mathematical competencies in kindergarten predict mathematics achievement through elementary, middle, and high school (Claessens, Duncan, and Engle 2009; Duncan et al. 2007; Watts et al. 2014). National data from the Early Childhood Longitudinal Study show that students who exit kindergarten below the 10th percentile in mathematics have a 70 percent chance of staying below the 10th percentile in fifth grade (Morgan, Farkas, and Wu 2009). Among low achievers, those most at risk are kindergartners who are low SES, female, African American and/or have learning-related behavior problems (Morgan, Farkas, and Maczuga 2011). Other IES-funded research suggests that boys have an advantage over girls in mathematics performance starting toward the end of kindergarten and widening in subsequent elementary grades (Lubienski et al. 2013; Robinson and Lubienski 2011). Gaps in mathematics achievement among different racial/ethnic, socioeconomic, and language groups are substantial and persistent, with students who are living in poverty, learning English as a second language, or are Black or Hispanic having substantially lower mathematics performance.[8] Further, about 13 percent of all public school students receive special education services, a third of them because of a specific learning disability.[9]

These findings highlight the urgency of improving early mathematics education in the United States.

The Writing of This Synthesis

As part of IES's assessment of its work, it asks groups of eminent scholars to review peer-reviewed journal articles and book chapters that are products of IES-funded research in a specific area (e.g., reading, early childhood, mathematics). The task for each group of scholars is to synthesize what the field has learned from IES-funded research on the topic, and summarize the results for a general audience that includes policymakers and other stakeholders. The syntheses are not intended to be typical research reviews, which provide a grand overview of research in a field. Nor are they intended to be What Works Clearinghouse reviews. Rather, the task is to look across the research projects that IES has funded to determine what has been learned and where empirical and theoretical progress has been made as a result of IES funding, and provide suggestions for further research in order to improve education in our country.

Topics for IES syntheses are determined by IES staff members who review the overall research portfolio to identify topics with multiple projects that have been completed, from which peer-reviewed articles and book chapters have already been published. Nationally recognized researchers in the topic area are selected to co-author the research synthesis. IES identifies the research projects that are relevant to the topic, and its contractor gathers the peer-reviewed journal articles and book chapters that were produced under these projects relevant to the topic

[8] http://nces.ed.gov/programs/digest/d13/tables/dt13_222.10.asp
[9] http://nces.ed.gov/programs/coe/indicator_cgg.asp

being reviewed. IES and its contractors consult with project directors and principal investigators when appropriate in order to ascertain the relevance of the funded project to the topic of the synthesis and confirm that all peer-reviewed articles emerging from these projects are included. The authors meet several times with IES staff, either in person or via conference calls, to discuss the focus of the synthesis and identify organizing questions or themes.

The authors are given a relatively short deadline of 4 to 6 months to produce a draft document. Under the broad question of what has been learned from IES-supported research, the authors review the published research and organize the synthesis under topics or questions that reflect the work that has been published. The authors may also include non-IES research in the synthesis to provide the background or context for the IES-sponsored research or to describe the work on which IES research builds. The authors interact with and receive feedback from IES staff during the development of the research synthesis. However, the authors are responsible for the final product. They use their collective expertise to determine the foci of the written report, and the synthesis reflects the authors' expert judgment as to the strength of the evidence presented in the published work and the contribution of the reviewed articles and book chapters to the synthesis topic.

Before the research synthesis can be published, it is subjected to rigorous external peer review through the IES Standards and Review Office, which is responsible for independent review of IES publications. The authors then respond to the peer-reviewer comments and make appropriate revisions.

The focus of the present synthesis reflects the research on programs, practices, and policies intended to improve mathematics outcomes funded through IES's National Center for Education Research (NCER) and National Center for Special Education Research (NCSER). Unlike other syntheses that used a Technical Working Group to define the topics to be addressed in the document, the authors of the mathematics synthesis, who have great expertise in their field, determined the focus areas with input from IES. In reading this synthesis, readers should remember that it is not intended to be an overview of all existing research on mathematics education. The authors were asked to review those published articles or book chapters that had emerged from projects funded by IES through NCER (U.S. Department of Education 2013) and NCSER (U.S. Department of Education 2012). Specifically, they were asked to review articles from peer-reviewed journals and book chapters from funded projects that were published from January 1, 2002, to June 30, 2014. Thus, there is a great deal of ongoing research that is not represented in this synthesis because some grants are not yet at the stage in the research process where findings are in and summarized for publication. Note also that reports of IES-funded research that have not been subjected to the peer-review process in publication are not included in this review. Given that authors were asked to review only those peer-reviewed articles and book chapters that emerged from NCER- and NCSER-funded projects available at the time this synthesis was written, there likely are peer-reviewed articles or book chapters emerging from ongoing NCER- and NCSER-funded research relevant to the synthesis topic.

To help address this national need to improve mathematics education, IES has funded almost 200 research grants on mathematics learning and teaching between 2002 and 2013 through NCER and NCSER. This research has identified instructional strategies, practices, and materials for helping students learn mathematics, and developed approaches to teacher professional development for improving mathematics teaching.

The authors concentrated on mathematics learning and teaching from kindergarten to grade 12,[10] as mathematics teaching and learning in preschool were covered in a previous IES Synthesis report (Diamond, Justice, Siegler, and Snyder 2013). Approximately 80 of the K–12 mathematics learning and teaching grants funded by IES had peer-reviewed publications (journal articles, conference proceedings, and chapters) between January 1, 2002, and June 30, 2014, which were products of these projects. [11] These included publications that were in press as of June 30, 2014. Grants that did not have findings for publication at the time of review were not included in this synthesis.

The authors obtained an initial list of publications from the NCER *Publication Handbook* (2013) and the NCSER *Research Programs and Funded Projects: 2005-2012* publication list. The contractor contacted principal investigators of all the grants to identify newer publications that had not been included in the IES reports. The authors reviewed all publications. In forming the synthesized contributions, the authors focused on publications that reported students' mathematics knowledge as an outcome, and the evidence needed to support a contribution generally had to consist of more than one publication from one grant. Thus, the report is not a general synthesis of research in the field, but rather what has been learned to date through review of peer-reviewed publications of 69 IES-funded grants on the topic.

The authors included research that used a range of methods—from longitudinal studies looking at the predictors of mathematics achievement, to one-on-one assessment and intervention studies, to large randomized-control trials of instructional materials. Readers should keep in mind that if an instructional technique or student characteristic is correlated with current mathematics knowledge or predictive of future mathematics knowledge, this does not mean the relation is causal; other, unmeasured variables may account for the relationship. Experimental research that involves the random assignment of students to instructional condition and appropriate controls can be used to support a causal link from the instructional condition to

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[10] High school was not excluded from consideration in the development of this synthesis. However, no peer-reviewed research beyond the ninth grade was identified.

[11] NCSER first began funding research in 2006.

improved mathematics knowledge. However, when experimental evidence is provided, it does not mean that every student benefited from the instructional approach, but rather, students *on average* benefited. An instructional approach is rarely effective for every single student.

The synthesis is broken into two sections. The first section focuses on improving mathematics learning for students in elementary school and in the middle grades. It includes development and evaluation of assessments, instructional strategies, and instructional materials. The second section focuses on the development and evaluation of teacher professional development approaches.

Improving Mathematics Learning

Teachers are faced with myriad choices for how to teach their students mathematics. Does using concrete materials to illustrate the problem help? What is the best way to organize practice problems? How can technology be used to support learning? Rather than basing teaching on intuitions, past experience, or the latest fad, research is needed to identify more and less effective instructional strategies. IES has funded numerous grants to develop and test instructional strategies and instructional materials for improving students' mathematics learning. Results of IES-funded research are organized into two sections based on mathematics topics and general grade levels: (a) whole numbers, operations, and basic problem solving in elementary school, and (b) fractions and algebra in the middle grades. This synthesis includes research on students in general education settings as well as students with or at risk for learning difficulties and disabilities.

Whole Numbers, Operations, and Word Problem Solving in Elementary School

Contribution 1. Measures emphasizing number, number relations, and number operations reliably identify students who are at risk for mathematics difficulties or disabilities.

Teachers need measures that accurately predict which students will have trouble learning mathematics so that they can provide effective support and early intervention for kindergarten and first-grade students. Much of the mathematics research in early screening and prevention focuses on the concept of number sense, which includes core knowledge of whole numbers, number relations, and number operations. Weaknesses in number sense characterize many students with mathematics difficulties and most with diagnosed mathematics learning disabilities (Jordan, Fuchs, and Dyson 2015).

IES-funded projects developed and demonstrated the usefulness of measures for teachers to document kindergarten and first-grade students' numerical proficiency. Clarke, Chard, and colleagues developed a set of measures to screen kindergartners and first-graders who may be at risk for developing mathematics difficulties (e.g., Chard et al. 2005). Four measures at the beginning of kindergarten successfully predicted performance on a standardized mathematics achievement test at the end of the year: oral counting, number identification, quantity comparison (e.g., name the larger of two numerals), and finding a missing number in a sequence (e.g., 4, __, 6). Similarly, Ginsburg, Lemke, and colleagues (Hampton et al. 2012; Lee et al. 2012) and Fuchs and colleagues (Seethhaler and Fuchs 2011) developed measures of number knowledge that effectively identified students who will need special help in mathematics.

Contribution 2. Early number competencies are malleable and can be taught successfully to students with and without mathematics difficulties through targeted and conceptually driven instruction.

A growing body of IES-funded research shows that number sense competencies can be boosted in at-risk kindergartners and first-graders as well as in older students with moderate to severe learning disabilities. Clarke, Chard and colleagues (Chard et al. 2008) developed a kindergarten mathematics curriculum, Early Learning in Mathematics (ELM), aimed at providing high-quality instruction in general education settings. They designed ELM to build students' number sense and prevent future learning difficulties in mathematics. ELM promotes mathematics learning in students who are on track, and also reduces the gap between at-risk students and those with no identified risk. Each 30-minute lesson, carried out in a general classroom setting, provides instruction to all students in numbers and operations, geometry, measurement, and vocabulary. ELM incorporates instructional principles, such as explicit instruction, use of mathematics models to build conceptual understanding, and multiple and varied practice and review activities. In an initial quasi-experiment (a study design in which groups are created in a process that is not random) (Chard et al. 2008), students who received the ELM curriculum showed better mathematics achievement than those who received business-as-usual instruction, and the program received high marks from teachers. A follow-up experiment (Clarke et al. 2011) revealed that the mathematics achievement gains of ELM students who were identified as at risk for mathematics difficulties were greater than the gains of ELM students with no identified risks. The reader is reminded that these findings, as is true of all experimental findings, indicate that children *on average* benefited from the intervention, but it does not mean that every child benefited from the instructional approach.

Building on the whole class ELM kindergarten program, Clarke and colleagues (2014) conducted an experimental study of the efficacy of *ROOTS*, a specialized kindergarten intervention that focuses on developing understanding of whole number concepts and computational procedures, with IES funding. At-risk students who received the ROOTS program made more gains on mathematics achievement assessments than a group of at-risk peers who did not receive the program. Moreover, the ROOTS students made more achievement gains during the course of the study than a group of typically-achieving peers, thus reducing the mathematics achievement gap.

Also with IES funding, Baroody and colleagues (2012) developed a 9-month training program to help kindergartners learn simple arithmetic combinations. The researchers showed that number fact fluency is built most effectively through understanding of number relations and principles rather than by rote memorization (Baroody, Bajwa, and Eiland 2009; Baroody, Feil, and Johnson 2007). Participating kindergartners all had low performance on a standardized mathematics achievement test and were not fluent with even the most basic addition combinations. The program helped students learn simple calculation rules. For example, they learned the number-after rule, where adding one always leads to the next number in the counting sequence (Baroody et al. 2012). The experimental study found that students who were taught calculation principles performed better than students who received business-as-usual instruction, both on practiced and new addition combinations as well as on a more general mathematics achievement test. Learning basic mathematics rules helped at-risk learners attain fluency with simple combinations and helped them connect informal knowledge (e.g., knowledge of the count sequence) to formal arithmetic problems. Reasoning strategies become automatic over time and provide a meaningful foundation for learning number facts.

Bryant and colleagues showed the effectiveness of specialized interventions for first-graders with mathematics difficulties (Bryant et al. 2011). Like Chard, Clarke, and colleagues, the team developed an intervention to build number concepts and skills. For example, students solved problems with visual representations of number lines (see also Contribution 3) and were taught specific strategies, such as counting onward from a number to solve an addition combination. In an IES-funded experimental study, students who used these activities performed better than students who did not on a mathematics computation measure, but not on solving applied mathematics problems. Importantly, the program reduced the percentage of students who needed more intensive assistance requiring special education services.

IES-funded research also evaluated *Math Recovery*, an existing one-on-one tutoring program focused on numeracy for low-performing first-grade students. Certified teachers who have received at least 60 hours of professional development so they can continuously assess students' current thinking and engage them in tasks to advance in each of six aspects of arithmetical knowledge provide the tutoring. Students receive four to five 30-minute one-on-one tutoring sessions each week for approximately 12 weeks. In an experimental study, struggling students who had received tutoring did substantially better on a diagnostic measure tied to the tutoring intervention, and moderately better on standardized mathematics measures, than students who had not received tutoring (Smith et al. 2013). Tutoring was particularly effective for students who had the most to learn. However, a year later, no effects of tutoring were found on any measure. This study highlights that gains children make during targeted interventions may not persist over time after the intervention ends.

Browder and colleagues (2012) developed a specialized curriculum for older special education students with moderate to severe intellectual disabilities with IES funding. Their conceptual model contained four active elements: target early numeracy skills related to number and operations, use systematic prompting and feedback, vary instruction with story-based lessons, and provide embedded instruction in general mathematics classes. Third- through fifth-graders with intellectual disabilities made gains with this approach, and they even applied the skills to other types of problems. These basic skills can be used to help students with disabilities learn relevant grade-level content, such as finding areas; dividing sets; or gathering, recording, and interpreting data as well as improve their daily lives (Saunders et al. 2013).

Contribution 3. Incorporating activities with number lines reveals and supports students' knowledge of whole numbers.

Number sense includes a good understanding of the magnitude of numbers (e.g., understanding of the magnitude of symbolic numbers such as 8 and 85). The magnitude of numbers can be represented as spatial positions on a physical number line, with small numbers on the left and larger ones on the right. Physical number lines capitalize on evidence that both students and adults represent the magnitude of numbers on a mental number line (e.g., Case and Okamoto 1996).

How precisely students represent numerical magnitude, as revealed by their number line estimation accuracy, is foundational to mathematics learning. First, Siegler and colleagues' IES-funded research has shown that students' accuracy at placing numbers on a number line is closely related to their mathematics proficiency on a range of measures, including standardized achievement tests (Booth and Siegler 2006, 2008). Second, their experimental research indicates that improving students' accuracy in placing whole numbers on number lines improves their counting, numeral identification, and learning of simple arithmetic in preschool (Ramani and Siegler 2008, 2011; Ramani, Siegler, and Hitti 2012; Siegler and Ramani 2009) and their arithmetic learning in first grade (Booth and Siegler 2008) as compared to students who did a variety of activities not designed to improve their magnitude understanding.

Siegler and colleagues identified an effective way to improve students' understanding of number magnitudes: playing games that involve numbers in a linear order, such as Chutes and Ladders (Laski and Siegler 2014; Ramani and Siegler 2008; Ramani, Siegler, and Hitti 2012; Siegler and Ramani 2009). In these studies, kindergarten or preschool students who were randomly assigned to play the games over several days had better number sense than students who did other mathematics activities. When students play these games, it is useful for adults to guide their attention to numerical magnitude. For example, adults should have students count on from their current space (e.g., if advancing two spaces from 11, counting 12–13, not 1–2), so they are practicing with the full range of numbers (Laski and Siegler 2014). Having a real-world context, such as a race course, also supported number line estimation accuracy compared to not providing a real world context (Saxe et al. 2013). Digital games that incorporate number line activities to support whole number knowledge have also been developed with IES funding (Ginsburg, Jamalian, and Creighan 2013; Ginsburg et al. 2015).

Contribution 4. Working memory capacity and computational fluency predict word problem-solving accuracy in the early grades.

Consider the following word problem: "Bart and Melissa are making cupcakes for a school fundraiser. Melissa's recipe calls for 3 more tablespoons of flour than does Bart's recipe. If Bart's recipe calls for 5 tablespoons of flour how many tablespoons of flour are in Melissa's recipe?" The capacity to attend to the information, hold it in short-term memory, process its meaning, and perform a numerical operation, facilitates word problem-solving accuracy (Swanson and Beebe-Frankenberger 2004).

In their IES-funded work, Swanson and colleagues (e.g., Swanson 2011; Swanson, Jerman, and Zheng 2008, 2009; Swanson, Kehler, and Jerman 2010; Zheng, Swanson, and Marcoulides 2011) examined the influence of "working memory"[12] in solving math word problems in students from first through third grades in correlational studies. As students grew in their working memory capacity, they also became more effective at solving word problems. Strong arithmetic computation skills seemed to help students compensate for working memory weaknesses, suggesting that both skill fluency and working memory contribute to word problem-solving proficiency in mathematics. If a child can compute quickly with little effort, the burden on working memory can be reduced for solving word problems.

Contribution 5. Training in how to use learning strategies improves word problem-solving skills in at-risk learners, although response to training may vary according to a student's working memory capacity.

A student's response to training in word problem-solving strategies depends on his or her working memory capacity (Swanson 2014). In an IES-funded experimental study, students identified as at risk for mathematics difficulties were randomly assigned to one of three strategy training groups or to a fourth group that did not receive any training. In the verbal strategy group, teachers taught students to find the question in a word problem, underline it, circle key words, and cross out irrelevant information. In the visual strategy group, teachers taught students to use visual diagrams to represent the word problems. Finally, in the combined verbal and visual strategy group, teachers taught students to use the verbal steps as well as visual diagrams. Overall, the findings showed that a student's level of working memory skill moderated the effectiveness of the cognitive strategy training. At-risk students with better working memory capacity benefited more from all of the strategy training approaches than did at-risk students with lower memory capacity. The most beneficial condition for the relatively high working-memory children was the one that emphasized only visual strategies. Surprisingly, cognitive strategy training actually decreased word problem-solving accuracy in children with lower working memory, challenging the assumption that strategy training helps at-risk children compensate for working memory deficits. This study, along with other related IES-funded studies from the research team (e.g., Swanson, Lussier, and Orosco 2013; Swanson et al. 2014; Swanson, Orosco, and Lussier 2014), suggests that not all students with or at risk for mathematics disabilities benefit from word-problem strategy training in mathematics. Limited working memory capacity seems to be a major barrier to the effectiveness of word problem-solving interventions, a finding also supported by Fuchs et al. (2014; see Contribution 12).

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[12] Working memory capacity refers to the amount of information individuals can hold and use in short-term memory.

Contribution 6. Dynamic assessments involving teacher-student interaction may improve assessment accuracy and instruction for students at risk for mathematics difficulties.

Dynamic assessment examines the amount of teacher support needed to learn unfamiliar or new mathematics topics, and thus has direct relevance to instructional planning. Dynamic measures contrast with conventional measures wherein students must respond without help (Tzuriel 2001). In IES-funded research, Fuchs and colleagues (Seethaler et al. 2012) tested the dynamic approach for predicting first-grade achievement in calculation and word problems. At the beginning of first grade, teachers assessed students with a dynamic tool that provided support for learning new mathematics topics, along with traditional or static mathematics and cognitive measures. The dynamic assessment significantly predicted students' performance at the end of the year on calculations, and especially on word problems, over and above the traditional measures. The findings suggest that the most effective assessments not only examine students' level of performance but also the amount and type of scaffolding needed to learn new concepts and skills.

In other IES-funded work, Swanson's team developed dynamic strategic mathematics (DSM) to help Latino English Learners (ELs) develop word problem-solving proficiency (Orosco et al. 2011). ELs may struggle in reading and language as well as in mathematics, and thus represent a high-risk population. They based the approach on the principle that instructional feedback can change a student's behavior; that is, a child can move from weak to better performance when the teacher modifies the tasks to provide clarity. DSM uses a variety of instructional techniques to move ELs progressively to stronger understanding of word problems. These techniques include helping students find the question and key words and numbers, set up the problem, solve the problem, and check for accuracy. In an IES-funded experimental study that used a single-subject design, the DSM strategy improved all students' performance on increasingly challenging word problems, and the improvements were maintained over time. The students practiced the formal language of mathematics word problems and used their English skills to solve the problems successfully.

Contribution 7. In contrast to popular belief, building blocks, "play money," and other manipulatives sometimes have limited value in teaching elementary school mathematics. Manipulatives and materials that have minimal visual distractions can be more effective than ones that are more realistic or complex.

Educators often use concrete manipulatives (e.g., blocks, tiles, figurines, play money), as tools to help students learn mathematics. For example, nationally, over 90 percent of kindergarten teachers reported using manipulatives for mathematics instruction (Bottia et al. 2014). Indeed, a recent meta-analysis indicates that instruction that incorporates concrete manipulatives can aid student learning better than instruction using only abstract math symbols (Carbonneau, Marley, and Selig 2013). However, they identified multiple studies that reported negative effects of instruction with concrete manipulatives. Echoing this concern, several IES-funded research projects suggest that teaching with concrete materials may not always promote mathematics achievement, and suggest ways that concrete materials can be chosen and used more effectively. How often teachers report using manipulatives is not related to mathematics achievement in kindergarten or first grade for most students (Bottia et al. 2014). Similarly, Morgan, Farkas, and Maczuga (2015) found that the reported frequency of using manipulatives in first grade was not related to mathematics achievement gains. The IES-funded studies were correlational and provide no information on how manipulatives were chosen or used. However, the research does suggest that current uses of manipulatives in instruction may not always aid mathematics learning.

IES-funded experimental research highlights the difficulty students have transferring skills they learn with manipulatives to solve problems without manipulatives (as is typical on tests). Uttal and colleagues investigated the effectiveness of using commercially available manipulatives (Digi-Blocks) to teach multidigit subtraction compared to using only written numerals (Uttal et al. 2013). The teaching approach included using written symbols to represent the actions taken with the Digi-Blocks. Although the rising second-grade students learned to solve subtraction problems using the manipulatives over a 2-day lesson, this skill did not transfer to solving written problems without manipulatives.

Experimental research also highlights features of manipulatives and other materials that can reduce their effectiveness. In particular, more realistic and visually complex materials may draw students' attention to the materials themselves rather than to the abstract numerical concepts they represent (Kaminski, Sloutsky, and Heckler 2008, 2009a, 2009b; McNeil and Jarvin 2007; McNeil and Uttal 2009). In support of this claim, IES-funded research by McNeil, Uttal and colleagues (2009) found that fourth- through sixth-grade students made more errors when solving word problems involving money if they were given realistic bills and coins to help them solve the problems compared to no money or "bland" bills and coins that had only the monetary value written on them. Accuracy was similar when given bland money or no money, suggesting that it was not the presence of manipulatives per se that hindered performance but rather that the realistic bills drew students' attention away from the abstract concepts. Similarly, Kaminski and Sloutsky (2013) found that kindergarten and first-grade students learned to read bar graphs better if the learning materials were shaded bars rather than bars that included discrete countable objects in their IES-funded research. When using the more concrete learning materials with discrete countable objects, students learned strategies that they could not apply to typical bar graphs that do not include discrete objects. In this study, there were no manipulatives; rather, the printed learning materials varied in how realistic and visually complex they were, suggesting a general principle not specific to physical manipulatives. Overall, extraneous information in concrete examples may distract the learner from the relevant mathematical structure, reducing transfer to new examples (Kaminski, Sloutsky, and Heckler 2008, 2009a). Using manipulatives and materials that have minimal visual distractions seems to help students focus on more general, abstract ideas that are central to the mathematics.

Contribution 8. Improving students' general reasoning skills may also improve their ability to learn mathematics.

Young students typically develop a range of reasoning skills that should support mathematical thinking, such as identifying patterns, recognizing similarities and differences, and reasoning about spatial relations. However, some students lag behind their peers in developing reasoning skills (Rittle-Johnson et al. 2013; Verdine et al. 2014).

Through IES-funded research, Pasnak, Kidd, and colleagues have tested approaches for improving students' general reasoning skills, with the goal of improving how much students learn from their regular mathematics instruction. For example, they predicted that identifying patterns (i.e., predictable sequences) was an important skill to have in order to learn from the first-grade math curriculum. Although many students develop pattern knowledge without targeted instruction, some know substantially less about patterns than their peers. In two IES-funded experimental studies, first-grade students with limited patterning skills were randomly assigned to learn about patterns, numeracy (e.g., counting, adding), reading, or social studies (Kidd et al. 2013, 2014). Students who received pattern instruction for 6 months performed as well or better on several standardized mathematics assessments relative to students who received numeracy instruction, and both types of instruction support better mathematics achievement than reading or social studies instruction (Kidd et al. 2013, 2014). In addition, students' pattern knowledge at the end of the intervention predicted their mathematics achievement, further supporting the importance of pattern knowledge for mathematics achievement. Similarly, three other experimental studies found that teaching kindergarten students general reasoning skills, such as identifying when one item in a set differs from others on a key feature, such as size or shape (i.e., oddity), supported greater mathematics achievement than literacy or art instruction (Kidd et al. 2008; Pasnak et al. 2008, 2009). Students were selected for intervention because they were weak in the target reasoning skills, and at the end of the school year, these reasoning skills had improved, and improvements in these skills predicted improvements in their mathematics knowledge. Directly supporting improvements in students' reasoning skills may help them learn better from classroom mathematics instruction.

Another important reasoning skill is spatial reasoning, defined as the ability to imagine and mentally manipulate spatial information (e.g., to imagine what an object will look like when rotated). IES funding supported a synthesis of past research on improving spatial reasoning, confirming that it can be improved with training (Uttal et al. 2013). Further, IES-funded research by Cheng and Mix (2014) found that providing 40 minutes of spatial training to 6- to 8-year-olds improved their calculation accuracy immediately after training, compared to a control group that solved crossword puzzles. Future research needs to establish whether training in spatial skills leads to sustained improvements in general mathematics skills.

Overall, this research highlights the importance of reasoning skills, including patterning and spatial skills, for learning mathematics. Reasoning skills can be improved, particularly for students who lag behind their peers, which in turn can improve mathematics learning.

Contribution 9. Simple changes in the formatting of arithmetic problems can help improve students' understanding of the equal sign.

Understanding the equal sign is critical for supporting future success in algebra (Knuth et al. 2006). Unfortunately, the way that arithmetic problems are typically presented is thought to support a frequent and persistent misunderstanding of the equal sign (McNeil 2014). Arithmetic problems are typically presented in an operations-equal answer problem format (e.g., 9 + 4 = __) and sequenced by the first addend (e.g., learning the 2's facts of 2 + 1, 2 + 2, 2 + 3, etc.).

IES-funded research by McNeil and colleagues has found that altering how arithmetic problems are presented can improve understanding of the equal sign in addition to fact fluency. In three experimental studies, practicing arithmetic facts in nontraditional formats improved second- and third-grade students' knowledge of the equal sign relative to traditional practice formats or no practice (McNeil et al. 2011; McNeil et al. 2012; McNeil, Fyfe, and Dunwiddie 2014). Nontraditional practice formats included arithmetic facts with operations on the right (e.g., __ = 9 + 4), arithmetic facts organized around equivalent sums (e.g., facts that add to 7, such as 3 + 4 = 7, 5 + 2 = 7, and so forth), and replacing the equal sign with "is equal to" on some problems. The nontraditional practice formats supported greater knowledge of the equal sign and similar fact fluency than more traditional practice both immediately and 6 months after finishing the practice. Additional experimental research found that an explicit lesson on the meaning of the equal sign was much more effective when the equal sign was presented in less common contexts such as 28 = 28 rather than in the context of operations-equals-answer problems (McNeil 2008). Thus, simple changes to how arithmetic problems are presented support better understanding of the equal sign.

Contribution 10. A supplemental mathematics curriculum that integrates the knowledge, skills, and teaching approaches used by Alaska Native people improves Alaskan students' mathematics knowledge.

The Math in Cultural Context curriculum integrates into curriculum materials, Alaska Native cultural knowledge and pedagogical processes, such as creating artifacts and navigating by the stars. The pedagogical strategies include expert-apprentice modeling because that is a common teaching approach within the community. Ten modules have been developed for students in grades 2 through 7, with initial funding from other sources. IES supported an experimental study conducted in rural and urban areas of Alaska that provided causal evidence for the effectiveness of Math in Cultural Context second-grade units on numeracy (especially place value) and measurement (Kisker et al. 2012). Alaska Natives, including students from the Athabaskan, Inupiaq, Tlingit, and Yup'ik cultural groups, as well as Caucasian and other non-native Alaska students participated, with Alaska Natives comprising about half of the sample. Students randomly assigned to receive the Math in Cultural Context units gained greater knowledge of the target content than students receiving the existing curriculum and approach, including on a retention test, for both Alaska Native and non-native Alaska students. This is particularly important because American Indian/Alaska Native students lag behind all other ethnic groups in mathematics proficiency (Aud, Fox, and KewalRamani 2010).

Overall, these findings support the potential effectiveness of using culturally based curriculum for raising mathematics knowledge of students from within that culture as well as from other cultures.

Fractions and Algebra in the Middle Grades

Contribution 11. A constellation of processes influences fraction learning, including numerical magnitude understanding, arithmetic fluency, attention, memory, and verbal skills.

Many students have persistent problems with fractions during the course of their school careers. In an IES- funded study, for example, Siegler and Pyke (2013) found that by sixth grade, low-achieving mathematics students (i.e., those performing in the bottom third in mathematics achievement) lag far behind higher achieving mathematics students in their knowledge of fractions, and this gap increases substantially between sixth and eighth grades.

IES-funded projects show that both number-related and general cognitive and behavioral processes influence early and later fraction learning (Seethaler et al. 2011). In a longitudinal study, Jordan, Hansen, et al. (2013) examined the degree to which general cognitive and behavioral processes and number-specific knowledge in third grade predict fraction knowledge at the end of fourth grade, right after students finished their first year of formal instruction on fractions. Overall, the ability to place whole numbers on a number line from 0 to 1,000 in the third grade was the most influential predictor of fraction outcomes in fourth grade. Seeing that both whole numbers and fractions have magnitudes that can be represented on number lines is a unifying insight that helps students progress in mathematics (Siegler and Lortie-Forgues 2014; Siegler and Lortie-Thompson 2014; Siegler, Thompson, and Schneider 2011). Other important predictors were arithmetic fact fluency, classroom attention, vocabulary skills, and working memory. Nonverbal proportional reasoning also predicts learning of fraction concepts (Hansen et al. 2015). In another IES-funded longitudinal study, findings suggest that general cognitive and behavioral predictors support students' learning of whole number skills in second grade, which in turn helps students learn fractions in fourth grade (Vukovic et al. 2014).

Contribution 12. Students with and without mathematics difficulties make greater gains when mathematics instruction emphasizes fractions as magnitudes that can be represented on a number line.

Students need to learn to think of fractions as magnitudes, not merely meaningless symbols (Behr et al. 1984; Siegler, Thompson, and Schneider 2011). With IES funding, Fuchs and colleagues developed and tested an instructional intervention, Fraction Face-Off!, which emphasizes placing and comparing fractions on number lines to support students' understanding of fractions as magnitudes. For example, the number line helps students see why $\frac{1}{4}$ is smaller than $\frac{1}{2}$ and why $\frac{3}{6}$ is equivalent to $\frac{4}{8}$ or $\frac{5}{5}$ is equivalent to 1. Participants in IES-funded research on the intervention included fourth-graders with learning difficulties in mathematics. In the first experimental study (Fuchs et al. 2013), students using the number line fractions intervention performed better on all fraction outcomes compared to students using a more conventional approach emphasizing parts of a whole. A subsequent experimental study (Fuchs et al. 2014) examined two variations on the use of the number line; one provided extra activities to build fraction understanding, while the other provided activities to build speed or fluency on the fractions topics that were taught. As expected, students in both number line fraction intervention groups performed better than their peers who did not receive the number line intervention. However, students' response to the two fractions interventions varied according to their cognitive skills. Students with relatively weak working memory capacity learned better with the extra conceptual activities, and students with relatively strong working memory learned better with the extra speeded fluency activities. Overall, the Fuchs team shows that an approach to teaching fractions that emphasizes number lines is much more beneficial than instruction that emphasizes parts and wholes. Moreover, within the number line approach, instruction can be enhanced further through adjustments based on individual differences in cognitive abilities, such as working memory. A discussion of the importance of working memory capacity to mathematics word problem solving can also be found under Contributions 4 and 5.

Saxe and colleagues also developed a supplemental curriculum unit on fractions as well as integers that emphasized placing and comparing numbers on number lines. In the Learning Mathematics Through Representation (LMR) curriculum, teachers use the number line as a unifying representation across whole numbers, integers and fractions, and students learn to construct, apply, and analyze the implications of mathematical definitions and concepts. Each of the 19 lessons follows a five-phrase structure, including differentiated partner work to address the needs of diverse learners. In an IES-funded experimental study, diverse fourth- and fifth-grade students who used the LMR curriculum had substantially higher knowledge of fractions and integers at the end of the intervention, and 5 months after its conclusion, than students who used another evidence-based curriculum (Saxe, Diakow, and Gearhart 2013; Gearhardt and Saxe 2014). Thus, for both students with learning difficulties and students in general education classrooms, instruction that emphasized fractions as magnitudes through the use of number lines promoted better fraction knowledge than more conventional instruction.

Contribution 13. Adolescents with mathematics difficulties benefit from fractions instruction that builds fractions skills and concepts alongside problems anchored in everyday contexts.

Focusing on fraction concepts, IES-funded research by Bottge and colleagues (e.g., Bottge et al. 2010a; Bottge et al. 2014) demonstrated the impact of enhanced anchored instruction (EAI) for helping adolescents with mathematics difficulties develop skills in solving fraction word problems. EAI activities center on an anchored problem and provide opportunities to practice skills through solving related problems in hands-on contexts (e.g., building skate board ramps to apply fraction operations and measurement) (Stephens, Bottge, and Rueda 2009). Teachers typically use EAI in general education mathematics or technology education classrooms (Bottge et al. 2010b).

In early work with EAI, the research team focused mainly on how to solve problems involving fractions while providing computational help informally on an as-needed basis. Middle school students made gains in problem solving but not in basic skills (Bottge et al. 2007). A revised version included a computer-based learning module designed to build basic fraction skills (e.g., finding fraction equivalences on a number line, adding and subtracting fractions with like and unlike denominators) more formally alongside the practical EAI problems. Bottge and colleagues (2010a) showed that while both the traditional EAI and the hybrid approach improved real world problem-solving skills with fractions (e.g., determining how much wood is needed to build a book case) at about the same rate, the revised approach that combined formal skills instruction with EAI problem solving gave students an added advantage in fraction computation and procedures. The findings from this and related reports (Bottge et al. 2010b; Bottge et al. 2014) suggest that fraction teaching that integrates help in fraction skills with instruction in real-world problem solving is most effective.

Contribution 14. Practice problems should be interleaved so that problems of different types are mixed together rather than grouped together by problem type.

Simple changes to how teachers and textbooks sequence practice problems can substantially improve learning and knowledge retention. Practice assignments typically have students solve a set of problems that are mostly from the day's lesson (i.e., blocked practice). In contrast, on exams, problems of different types are mixed together. Decades of basic research on memory and learning suggest that practice problems need to be interleaved, so that problems of different types are mixed together (see Rohrer 2012 for a review). Interleaved practice also naturally leads to greater spacing of practice on each problem type, which greatly improves retention of the information (see Carpenter et al. 2012 for a review).

In IES-funded research, Rohrer and colleagues confirmed dramatic benefits of interleaved practice on mathematics homework assignments in the middle grades (Rohrer, Dedrick, and Burgess 2014; Rohrer, Dedrick, and Stershic in press). In both experimental studies, seventh-grade students completed 10 practice assignments over several months on mathematics topics such as graphing linear functions and finding slope. Accuracy was dramatically higher on the problem types with interleaved practice rather than blocked practice (e.g., 74 percent vs. 42 percent correct when tested 1 month later). Note that the amount of practice was the same; only the sequencing of practice problems differed.

Interleaving practice problems is one of four research-based principles of learning that an IES-funded Research and Development (R&D) center used to redesign a widely implemented middle-school mathematics curriculum. An initial IES-funded experimental evaluation of four of the revised curriculum units indicates that students who were taught using the revised units had greater knowledge of the target content on an immediate posttest than students who were taught using the original curriculum (Davenport, Kao, and Schneider 2013). This project illustrates that interleaved practice can be incorporated throughout the curriculum.

Contribution 15. Comparing multiple ways to solve problems improves student learning, and teachers can help students make effective comparisons.

Instruction that emphasizes comparisons—i.e., reflecting on how two examples are similar and different—may support learning in a variety of tasks and with various age groups. Comparing solution methods is also a recommended instructional practice for mathematics education, based on case studies of expert teachers (National Council of Teachers of Mathematics [NCTM] 2000). IES-funded research has studied the use of comparison in mathematics instruction and ways to effectively support comparison in the middle grades.

Rittle-Johnson and Star conducted a series of IES-funded experimental studies to evaluate the impact of comparing solution methods for learning about computational estimation and equation solving. Students worked with a partner to compare examples of different solution methods for solving the same problem or studied the same examples presented one at a time; all students responded to reflection questions orally and in writing. Students who compared solution methods were much more likely to comment on the efficiency and accuracy of different ways to solve the same problem (e.g., Rittle-Johnson and Star 2007). Comparing solution methods consistently supported procedural flexibility (i.e., knowledge and use of multiple methods to solve problems) for both topics. Comparing methods sometimes supported greater success in solving problems (Rittle-Johnson and Star 2007; Rittle-Johnson, Star, and Durkin 2009) or greater knowledge of concepts (Rittle-Johnson and Star 2009; Rittle-Johnson, Star, and Durkin 2009; Star and Rittle-Johnson 2009). However, comparing methods was less effective when students had little prior knowledge of the topic (Rittle-Johnson, Star, and Durkin 2009). For these students, the lessons needed to be slowed and cover less content (Rittle-Johnson, Star, and Durkin 2012). Overall, comparing methods helps students learn mathematics, but its advantages are more substantial if students have some prior knowledge of the topic.

Two other IES-funded research projects have described how teachers support comparison in classrooms, revealing specific behaviors and approaches teachers might use to help improve the effectiveness of comparisons in mathematics teaching. Richland and Holyoak have studied how eighth-grade mathematics teachers in the United States, Hong Kong, and Japan used comparison in the classroom (Richland, Holyoak, and Stigler 2004; Richland, Zur, and Holyoak 2007). Teachers in all three countries frequently used comparison in their instruction, but U.S. teachers provided much less support for students to process and learn from the comparisons (Richland, Zur, and Holyoak 2007). For example, U.S. teachers were less likely to have the two things being compared visible during comparison or put the information to be compared side by side. Alibali and colleagues have studied how skilled U.S. teachers use gesture to support comparison across a variety of grade levels. Skilled teachers used gestures to guide students' attention and support links between examples and ideas (Alibali and Nathan 2009; Alibali, Nathan, and Fujimore 2011). Thus, correlational evidence suggests that to support effective comparison, teachers should keep both examples visible, present the examples side-by-side, and use gestures to help students compare aspects of the examples (see Richland, Stigler, and Holyoak 2012).

Contribution 16. Critiquing common incorrect solutions improves student learning.

Students have common and persistent misconceptions in mathematics. For example, when working with decimals, students often believe that 0.25 is larger than 0.5 because 25 is larger than 5. Previous theory and evidence from science learning suggest that students need to directly confront and reflect on misconceptions in order to overcome them (Eryilmaz 2002; Huang, Liu, and Shiu 2008; Van den Broek and Kendeou 2008). This stands in contrast to concerns by teachers that presenting and discussing incorrect solutions will reinforce and increase their use. Three IES-funded projects support the benefits of critiquing common incorrect solutions for mathematics learning in the middle grades.

First, Durkin and Rittle-Johnson (2012) found in an experimental study that fourth- and fifth-grade students learned more about decimals from an individual tutoring session when they compared examples of correct and incorrect solution procedures rather than only correct procedures. Second, building on this finding, McLaren and colleagues developed a web-based tutoring system for learning about decimals that included incorrect examples (Adams et al. 2014). When shown how a fictional student had incorrectly solved a problem, students had to identify what the student had done wrong, correct the mistake, and explain why the new answer was correct. Students also practiced solving similar problems. Students in the control condition solved all of the problems and explained why the correct answers were correct. In this experimental study, sixth- and seventh-grade students who used the version with incorrect examples had greater knowledge on a 1-week retention test, suggesting that reflection on incorrect examples supported better retention of correct knowledge. Similarly, Booth and colleagues integrated correct and incorrect examples into a commercial, evidence-based intelligent tutoring system for equation solving (i.e., Algebra I Cognitive Tutor; Booth et al. 2013). Students who were randomly assigned to work on the tutoring system with correct and incorrect examples gained greater knowledge of algebra concepts than students who only solved problems. Booth and colleagues have also incorporated correctly and incorrectly worked examples into written homework and in-class assignments, and found improved learning (Lange, Booth, and Newton 2014). Finally, this principle of learning is one of four research-based principles of learning that has been used by an IES-funded R&D center to redesign a widely used middle-school mathematics curriculum that show promise for improving the effectiveness of the curriculum (Davenport, Kao, and Schneider 2013).

Contribution 17. Promoting fluency in mapping between different representations of mathematical ideas such as matching a word problem to an appropriate number sentence improves students' learning.

IES-funded research has supported the development of promising instructional strategies that help students link different representations of mathematical ideas. The research has focused on fractions, equations, and linear functions. Kellman and Massey have developed exercises in which learners recognize, discriminate, and map across representations of mathematical ideas in many short practice trials. For example, in fraction exercises, students match one representation of a fraction problem to another representation, such as matching a word problem to an appropriate number sentence or picture (Kellman et al. 2008). Learners do not perform calculations or solve problems. In algebra exercises, students match linear functions represented in graphs, equations, and word problems (Kellman, Massey, and Son 2010) or match transformations of equations (Kellman et al. 2008). Adaptive computer software adjusts the exercises based on student success until they master the material. In a series of IES-funded experimental studies, middle- and high-school students who completed these matching exercises were more successful at solving related mathematics problems than students who did not complete the exercises (Kellman et al. 2008; Kellman, Massey, and Son 2010).

Contribution 18. Producing physical movements and gestures may improve students' mathematical learning.

There is increasing evidence that ideas, including mathematical ideas, are not just remembered in the abstract, but are grounded in body movements, often called embodied cognition (Gibbs 2006). Indeed, IES-funded research by Alibali and Nathan suggests a potential role of body movement, particularly gestures, in mathematical thinking. In particular, Alibali and Nathan (2012) argued that gestures reveal how both teachers and students mentally simulate actions and link to experiences with the body when talking about mathematics. For example, teachers used gestures to simulate actions, such as placing their arm at different angles to simulate the action of altering the slope of a line.

Based on observational research such as this, Landy and colleagues have used IES funding to develop exercises in which learners physically move symbols in algebraic expressions. For example, in their *Algebra Touch Research* software, students group like terms in an algebraic expression by touching appropriate symbols and moving them into the desired location (e.g., simplifying the expression $-5 + 4x + 2 + -6x + -8y + 2y$ by moving the $4x$ on top of the $-6x$). The software does not allow students to make mistakes, so students receive immediate feedback on algebraic transformation rules. Initial promise for the approach was provided with a group of eighth-grade students, although there was no control group or evidence for more general benefits to algebraic competence (Ottmar, Landy, and Goldstone 2012). More broadly, they have developed a new theory of symbolic reasoning that emphasizes motor and perceptual experiences and proposes principles that guide symbolic reasoning (Landy, Allen, and Zednik 2014).

Contribution 19. Teaching students cognitive strategies for solving word problems, such as categorizing, supports their word problem-solving success more than typical classroom instruction.

With IES support, Jitendra and Star developed a curriculum unit on ratio, proportion, and percent word problem solving called Schema-Based Instruction (SBI). Students learn to categorize word problems based on their mathematical structure, use diagrams to represent the mathematical relations between quantities in the problems, choose among multiple strategies to solve the problems, and monitor their success implementing the strategies. Across three experimental studies, seventh-grade students in the SBI condition had much greater success solving proportional word problems than students receiving typical classroom instruction, both immediately and several months later (Jitendra et al. 2009; Jitendra et al. 2011; Jitendra, Lein, et al. 2013; see also Jitendra, Star, et al. 2013). However, students in the SBI condition did not have greater success on a transfer test assessing success on novel problems such as probability or on the state achievement test. A fourth experimental study focusing on SBI for word problems involving percent found improvements in high-achieving but not low-achieving classrooms (Jitendra and Star 2012). SBI was designed with particular attention to the needs of struggling learners, but low-achieving students may need extended time to not only identify the underlying problem structure to adequately represent information in the problem, but also learn to flexibly choose among multiple strategies.

In other IES-funded research, Montague and colleagues tested the efficacy of Solve It! (Montague 2003), a word problem-solving approach for students with mathematics difficulties. Solve It! aims to teach struggling students how to analyze, solve, and evaluate mathematics word problems in ways similar to those described by the Swanson research team for younger students (Contribution 5). The focus is on developing effective strategies for solving word problems more generally (Krawec and Montague 2012). Students are taught how to paraphrase the problem in their own words, translate the word problem into a visual representation that shows the relations among its parts, develop a problem-solving plan, and check work for mistakes, among other cognitive strategies. Studies show that Solve It! is effective for most students in middle school, especially those with learning disabilities and low achievement (Krawec et al. 2013; Montague, Enders, and Dietz 2011). In a quasi-experimental study with eighth-graders (Montague, Enders, and Dietz 2011), students who received the intervention in general education classes, including those with identified learning disabilities, showed more growth in word problem solving than students who did not. The findings were replicated in a study with seventh-graders, with the intervention effects stronger for low-achieving and average-achieving students than for high-achieving ones (Montague et al. 2014).

Contribution 20. Using computer-based interim assessments provides teachers with diagnostic information and can improve students' mathematics achievement.

Appropriate instruction requires that teachers know their students' strengths and weaknesses and adjust their instruction accordingly. Interim computer-based assessment programs have been used by teachers and schools to monitor student progress toward state standards with the goal that teachers identify weaknesses in individual students' knowledge and adjust their instruction as needed. Many states and school districts are investing in such programs as part of an effort to improve performance on high-stakes state tests (Davidson and Frohbieter 2011).

With IES funding, Miller and colleagues evaluated the impact of a computer-based interim assessment system adopted by the state of Indiana (Konstantopoulos, Miller, and van der Ploeg 2013; Williams et al. 2014). Researchers randomly selected schools to implement the interim assessment system and third- through eighth-grade students in those schools scored substantially higher on the state achievement test in mathematics and in reading than students at schools that did not use the system. However, kindergarten through second-grade students in those schools did not score higher on a standardized test selected by the researchers. The studies suggest that interim assessment systems may be a promising tool for improving student scores on state achievement tests in grades 3–8. Additional research is needed on their effectiveness in younger grades.

Traditional assessments capture only whether students answer questions correctly or incorrectly. However, students often make systematic errors that reflect persistent misunderstandings that need to be addressed (Booth et al. 2014). A separate IES-funded project worked to develop a computer-based interim assessment for algebra I. Russell and colleagues developed a computer-based Diagnostic Algebra Assessment System (DAAS) for components of algebra I. The assessment system generates a teacher report for each student that includes both the percentage of problems solved correctly and the percentage of responses reflecting common misconceptions (i.e., persistent incorrect ways of thinking). For each topic, the research team developed two supplemental lesson plans for teachers to use if a misconception was diagnosed in their classroom. In an experimental study, students performed better on an algebra assessment if their teachers used the DAAS rather than interim assessment only (Russell, O'Dwyer, and Miranda 2009). This research, although preliminary, suggested that interim assessments that document misconceptions and include supplemental lessons for addressing diagnosed misconceptions may be more effective than assessments that only report on student accuracy.

A third IES-funded project found that state tests can be used to assist teachers in diagnosing specific problem areas for individual students. Using advanced measurement models, Embretson and Yang (2013) examined the sources of poor performance on a statewide eighth-grade achievement test. A group of students who performed just below the overall proficiency cut point varied considerably in how they performed in different component areas, and clear patterns of mathematics competencies emerged for individual students. For example, in the number/computation component area, the model identified specific attributes that underlie student difficulties, with the lowest level being knowledge of the order of operations and the highest level being knowledge of different number systems. Such information could help teachers differentiate instruction to address individual student needs, although this has not been directly evaluated.

Contribution 21. Computer-based tutoring systems can allow for individualized mathematics instruction and have the potential to help students learn mathematics. Systems ranging from a year-long integrated algebra I curriculum to an online assignment system improve students' mathematics learning, although some tutoring systems for raising state test scores among low-income students have shown less promise.

Computer-based tutoring and assessment systems provide promising approaches to individualizing instruction (Woolf 2009). Computer-based tutoring systems allow students to receive immediate feedback and help, and to work at their own pace, mastering one topic before moving on to the next. Both computer-based tutoring and assessment systems also provide teachers with timely feedback on individual student performance.

IES funded a large-scale evaluation of Cognitive Tutor Algebra I, which is a commercially available first-year algebra course that integrates a paper curriculum with an intelligent tutoring system that includes customized help and problems based on students' prior performance. The evaluators randomly assigned a large number of middle and high schools to use the curriculum or continue with their usual instruction. Teachers in the Cognitive Tutor group received the typical professional development and support provided by the publisher. In the second year of implementation of the program, students in the Cognitive Tutor group had higher scores on a standardized algebra assessment than students receiving typical instruction (Pane et al. 2014). The benefits were primarily for high school rather than middle school students and were not present in the first year of implementation. Implementation of the Cognitive Tutor Algebra I curriculum under normal conditions has promise for raising algebra achievement among high school students, although these benefits may not be present in the first year of implementation.

IES also funded Heffernan, Koedinger, and colleagues to develop ASSISTments, a free web-based system that provides assistance in the form of feedback and hints for solving mathematics problems; it also generates a report for teachers so that they know what topics were particularly challenging for their students. Teachers can select from existing problem sets on a wide range of mathematics topics for grades 4–8 or enter their own problems. ASSISTments is used by thousands of teachers across the United States and shows promise for improving students' state-test scores and learning from classwork and homework, as well as providing teachers with good predictions of student performance on statewide tests. For example, on the end-of-year state test, students at middle schools that used the ASSISTments system for some class assignments outperformed students at a comparison school that did not use the system, and the effect was strongest for special education students (Koedinger, McLaughlin, and Heffernan 2010). Performance on the ASSISTments system also predicted students' scores on the end-of-year state test (Anozie and Junker 2006; Feng, Heffernan, and Koedinger 2009). ASSISTments can also be used for homework assignments. In two experimental studies, students made greater gains when they completed their homework on ASSISTments rather than as typical paper assignments (Mendicino, Razzaq, and Heffernan 2009; Singh et al. 2011). Overall, ASSISTments is a promising computer-tutoring system for assigning classwork, homework, and test practice that provides feedback to both students and teachers.

IES has also supported the development and refinement of three computer-based tutoring systems for particular mathematics topics. Each provides individualized instruction by sequencing problems based on student performance, working on each skill until mastery, and has extensive help resources. They also incorporate evidence-based features such as support for self-monitoring and strategy selection. For example, Wayang Outpost provides strategy instruction and practice for solving geometry, statistics, and algebra problems that commonly appear on standardized tests (Arroyo, Woolf, and Burleson 2011). AnimalWatch supports solving word problems in the context of tracking and monitoring the status of endangered species, and providing opportunities for practicing arithmetic and fraction computations in real-world contexts (Beal et al. 2010; Cohen, Beal, and Adams 2008). On SimStudent, students tutor a computer avatar (SimStudent) on algebra by posing problems to solve and providing feedback and hints (Matsuda et al. 2013a; 2013b). For each tutoring system, there is some preliminary evidence that the system improves student learning of the target content.

Finally, IES also funded an experimental evaluation of the efficacy of two existing computer-tutoring systems for improving state test scores in low-performing schools. One, Spatial-Temporal (ST) Math, is a commercially available supplemental computer-tutoring system designed to help students in grades 2–5 visualize mathematics concepts and link them to problem contexts using game-like exercises. Students at low-performing schools randomly chosen to use ST Math scored better in one of five content areas on the end-of-year state test (understanding of whole number place value), but did not score better overall (Rutherford et al. 2014; Schenke, Rutherford, and Farkas 2014). Classroom observations and teacher surveys suggested that ST Math did not change classroom teaching practices or teacher beliefs, contrary to expectations (Tran et al. 2012). The other is a computer-based K-5 supplemental curriculum that uses brief multimedia lessons followed by exercises with immediate, individualized feedback (Suppes, Holland, Hu, and Vu 2013). Students at low-performing schools randomly chosen to use the supplemental curriculum did not score better on the end-of-year statewide exam, although there was some indication that more frequent use of the curriculum was associated with better test scores. Evaluations of these computer-tutoring systems are notable because they involved low-income students attending struggling schools, used random assignment of schools to use the computer-tutoring system or continue with business-as-usual instruction, and used a general outcome measure of great importance to schools. However, they had very limited success in raising state test scores among low-income students.

Contribution 22. Using technology to support student collaboration may improve students' learning.

When students collaborate, they have opportunities to give and receive help and share knowledge and resources. IES-funded research suggests there is a correlation between mathematics achievement and more frequent use of collaborative activities in the classroom (Bottia et al. 2014), especially more frequent engagement with other students' ideas during collaboration (Webb et al. 2014). IES has supported the development or evaluation of four technology-based tools for supporting student collaboration, described below.

First, Roschelle and colleagues developed a technology-mediated, Peer-Assisted Learning (TechPALS) approach. In TechPALS, a set of collaborative group activities using handheld devices support student practice with initial research focusing on fraction learning (Roschelle et al. 2010). For example, on an opening consensus activity, each student in a group selects an answer to a problem on his or her handheld device, and then the group must work together to reach a consensus until they all choose the correct answer. TechPALS can also be used in formative assessment, providing real-time feedback to teachers on how groups are performing so that teachers can provide assistance to struggling groups or modify their own future instruction to focus on a particularly difficult problem. In an experimental evaluation in three schools, fourth-grade students participated during a fraction unit (Roschelle et al. 2010). After teacher-led instruction, students practiced in small groups using TechPALS or individually on computer software. Students in the TechPALS condition showed greater fraction knowledge at the end of the unit and were more likely to ask questions, give explanations, and discuss disagreements with other students, all indicators that TechPALS increased collaborative learning. Whether learning gains would persist over time or be effective for other mathematics topics is an open question, but the findings supported the promise of TechPALS as a technological approach to supporting collaborative learning and formative assessment. Second, Aleven and colleagues have also worked to develop technology to support collaborative learning using IES funding. They modified a computer-based tutor on fractions to support collaborative learning, with some preliminary evidence supporting the promise of the approach relative to working on the tutor individually (Belenky et al. 2014; Olsen et al. 2014).

Third, Owens and colleagues studied the impact of using the TI-Navigator ™ system for promoting collaboration and learning in algebra I classrooms (Pape et al. 2013). The TI-Navigator system involves sets of handheld graphing calculators that are connected to a teacher's computer. Teachers can use the system to pose questions to groups of students, display a summary of students' (anonymous) responses, see screen captures of students' work to monitor progress and on-task behavior, and display screen captures so students' responses can be discussed. Thus, it should support student collaboration; formative assessment with immediate, nonthreatening student feedback; and classroom discussion of student work. In an experimental study, students who used the TI-Navigator system performed better on an algebra assessment at the end of the school year than students who used graphing calculators without connectivity (Pape et al. 2013). Teachers who used the TI-Navigator system often reported that the technology improved their understanding of their students' knowledge (Shirley et al. 2011). In summary, both handheld technology and computer-based tutors can support collaborative learning activities and provide formative assessment data to teachers, and these technologies have the potential to improve student learning.

Contribution 23. Improved synthetic speech shows potential to make algebra more accessible for students with visual impairments.

With IES funding, Frankel, Brownstein, and Soiffer 2013 developed ClearSpeak, an improved style of synthetic speech to help students with visual disabilities learn algebraic expressions in secondary school. Algebraic expressions, which use complicated visual representations involving nested parentheses, fractions, exponents, and so forth, are hard to access through audio channels. ClearSpeak "speaks" mathematics in a familiar way but with adjustment rules to make sure the spoken mathematics is not ambiguous (e.g., consistently referring to common fractions with a denominator of 20 or more as "*x* over *y*," use of "power" instead of "super"). The audio access can be used with onscreen visual access or with printed or braille documents. Preliminary studies by Frankel and Brownstein suggest that ClearSpeak boosts students' mathematics confidence and accuracy compared to older mathematics audio programs, although more research is needed to evaluate features of the ClearSpeak program.

Contribution 24. Increasing instructional time in algebra I for low-performing ninth-grade students can improve their course grades and test scores, but mandating algebra I by ninth grade may not improve test scores or college attendance.

In response to concerns that too few students are graduating from high school with the mathematics skills needed for college or the workforce (American Diploma Project 2004), some school districts have mandated college-preparatory mathematics coursework for all students and increased instructional time in algebra.

In IES-funded research, Allensworth and colleagues evaluated the impact of the Chicago Public Schools system's policies on mathematics coursework. First, they compared cohorts of students who attended Chicago public high schools before and after implementation of a district mandate that all students enroll in algebra I by ninth grade (Allensworth et al. 2009). Before implementation of the policy, about 80 percent of ninth-graders enrolled in algebra I; after implementation, almost all did. Among low-ability students, about 10 percent more earned credit for algebra I after implementation of the policy. However, students' scores on the end-of-year state test (which covered algebra and other types of mathematics) did not increase, nor did the likelihood that students would attend college. Further, state test scores for high-skill students declined (Nomi 2012).

Chicago Public Schools then mandated a *double-dose* of algebra for some students—all students below the national median on the eighth-grade mathematics test had to have two periods of algebra I a day in ninth grade. Again, Allensworth and colleagues compared cohorts of students before and after implementation of the mandate. Students who had a double-dose of algebra had higher scores on an algebra subtest of a standardized mathematics test and higher course grades in algebra and geometry (Nomi and Allensworth 2009). Student surveys indicated that students who had double-dose algebra reported more challenging content and more interactive instruction, with students explaining and discussing more (Nomi and Allensworth 2011). Students with above-average skills also showed higher standardized test scores after implementation of the policy, although *more* of these students received a failing grade in the course because they had been sorted into classes with higher-skilled students and their skills were low relative to classroom peers (Nomi and Allensworth 2013). In addition, the policy did not benefit students with low ability (below the 20th percentile).

Development and Evaluation of Teacher Professional Development Approaches

Improving student achievement may require changes in teaching practices. Most schools require teachers to attend professional development, and there are numerous mathematics teacher professional development programs on the market. However, in a review of teacher professional development programs targeted at improving K–12 mathematics instruction, only two programs were identified that had rigorous evidence for improving student mathematics outcomes (Gersten et al. 2014). In response to the need for evidence-based teacher professional development programs, IES has funded the development and evaluation of several teacher professional development programs.

Contribution 25. Teacher professional development that helps elementary school teachers build on a previous Pre-K mathematics intervention can boost the long-term effectiveness of the Pre-K intervention.

A persistent issue with educational interventions is that their effects fade over time (Leak et al. 2012). In their IES-funded work, Clements and Sarama hypothesized that fading effects reflect the inadequacy of future educational contexts to build on the skills developed in the intervention. To test this claim, they provided professional development to a group of kindergarten and first-grade teachers on ways to build on knowledge that their students had developed in an evidence-based Pre-K mathematics intervention (the follow-through group). The researchers also asked these teachers to integrate computer-software activities from the Pre-K intervention into their curriculum. In an experimental study, among students who received the Pre-K intervention, those who had the follow through had higher mathematics scores at the end of first grade than those who did not have follow through, and both groups had higher mathematics scores than students who did not receive the Pre-K intervention (Clements et al. 2013; Sarama et al. 2012). One benefit of the follow through was an improvement in the classroom culture (e.g., teachers' interactions with and responsiveness to students). Most of the students were from low-income families, and the sustained benefits of a Pre-K mathematics intervention through kindergarten and first grade were encouraging.

Contribution 26. Teacher professional development using a Lesson Study approach and targeting specific mathematics content can improve teachers' knowledge and lead to improvements in students' learning of fractions.

One promising teacher professional development approach is based on the dominant form of teacher professional development in Japan, called *Lesson Study* (Lewis et al. 2012). In teams, teachers first study the curriculum, then plan a lesson, do the lesson, and reflect on student learning in order to improve it and draw out broader lessons (called a study-plan-do-reflect inquiry cycle). This approach has teachers attend carefully to student thinking and be leaders in improving their own instruction. With IES funding, Lewis and colleagues have worked to study and spread the use of Lesson Study in the U.S. Their initial effort was to design a summer institute for helping interested teachers engage in Lesson Study with intensive support from the research team (Takahashi, Lewis, and Perry 2013). In hopes of having a more sustainable form of teacher professional development, they then developed a Lesson Study resource kit. Teachers can use the resource kit to engage in Lesson Study at their schools without involvement of the researchers (Lewis and Perry 2014; Lewis et al. 2012). The resource kit focuses on fractions and includes suggested fraction tasks; examples of curriculum materials from Japan, including videos; and templates to support the Lesson Study process. In an initial evaluation of the Lesson Study resource kit for fractions, volunteer teams of elementary school teachers, many of whom had already engaged in Lesson Study, were randomly assigned to use the fraction resource kit or to a control group. Teachers who used the Lesson Study resource kit had greater fraction knowledge at the end of the study than teachers who had not (Lewis and Perry 2014). In addition, their students also developed greater fraction knowledge than students receiving business-as-usual instruction on fractions (Gersten et al. 2014). Future research is needed on whether the resource kit is sufficient to improve student learning when teachers have not had prior experience engaging in Lesson Study.

Contribution 27. Teacher professional development that helps teachers create a supportive and safe environment for learning can improve the quality of third-grade mathematics instruction, although it does not directly improve end-of-year state test scores.

The Responsive Classroom is a widely used teacher professional development program designed to help teachers create a supportive and safe environment for learning. It includes allowing students to choose from several approaches to conduct their work, modeling desired behaviors, using positive teacher language, and demonstrating logical consequences. The approach is meant to support the social interactions among students that are required for students to engage in discussions and build on the ideas of others (Elliott 1999).

In an IES-funded study, Rimm-Kaufman and colleagues experimentally evaluated the effectiveness of the Responsive Classroom teacher professional development approach among elementary school teachers who were required by their schools to use it (after schools were randomly assigned to incorporate the Responsive Classroom approach or not). They found that third-grade teachers in the Responsive Classroom schools used more high-quality mathematics teaching practices, such as soliciting explanations and justification from students and engaging in problem solving, than teachers in the comparison schools, and more frequent use of Responsive Classroom practices was associated with greater use of high-quality teaching practices (Ottmar et al. 2013). However, participation in the Responsive Classroom did not directly improve students' mathematics achievement on the end-of-year state test (Rimm-Kaufman et al. 2014). Moreover, students attending schools that used the approach for 3 years did not score higher in fifth-grade mathematics (or reading) achievement than students attending schools that did not use the approach. Recall that teacher participation was mandated by the school, and some teachers who received training failed to implement the method sufficiently. More frequent use of Responsive Classroom practices among teachers assigned to use the approach was associated with better achievement, suggesting that the

Responsive Classroom approach may improve student achievement when implemented as intended (Rimm-Kaufman et al. 2014).

Contribution 28. Insufficient support from principals and insufficient teacher knowledge are potential barriers to effective teacher professional development.

In two IES-funded projects, researchers identified several potential barriers to successful implementation of district-mandated teacher professional development for elementary- and middle-school teachers. When districts mandate teacher professional development, there is large variability in teacher and principal commitment to change. Teacher reports and correlational evidence suggest that support from the principal and other school administrators seems particularly important. When studying the Responsive Classroom approach as described in Contribution 27, Rimm-Kaufmann and colleagues found that elementary school teachers emphasized the importance of principal buy-in, and weaker principal buy-in was related to more limited implementation of the target teaching practices (Wanless et al. 2013). In other IES-funded research, Santagata and colleagues found that even though their teacher professional development for sixth-grade teachers on three topics (fractions, ratios and proportions, and expressions and equations was mandated, school administrators often co-opted professional development sessions for other purposes, leading to substantially less professional development time than intended [Givvin and Santagata 2011; Santagata et al. 2011]). Santagata and colleagues also reported substantive limits in teachers' understanding of the mathematics content, knowledge of their students' understanding, and ability to analyze students' work and reasoning, which appeared to limit the effectiveness of their teacher professional development (Santagata 2009). Because this research was correlational, no causal claims can be made, but the research suggests factors that should be taken into consideration when new professional development programs are implemented and evaluated.

Future Directions

Based on their review of published papers describing results of IES-supported research from grants awarded between 2002 and 2013, the authors identified 28 specific contributions to support mathematics learning from kindergarten through secondary school. The contributions were organized by topic and grade level: Whole numbers, operations, and basic problem solving in elementary school and fractions and algebra in the middle grades. Many more funded studies that did not have published results by 2013 are likely to produce additional findings on mathematics learning on these topics as well as on topics not addressed in this synthesis, such as mathematics learning in high school.

Innovative efforts to translate findings from the education sciences into effective classroom practice must continue through rigorous research in the schools and other educational environments. Future research should build on findings presented in this mathematics research synthesis in several ways. First, studies of promise or ones that demonstrate positive results must be replicated and extended to ensure that the findings can be reproduced in multiple authentic educational settings, improve student achievement on measures used by teachers and schools, and lead to improvements that are sustained over time. Moreover, future work should continue to innovate and test new strategies for improving mathematics achievement. Research should examine the features of interventions that most effectively build concepts and skills in mathematics topics and address whether observed gains can be transferred to other areas of mathematics learning. Finally, future research must continue to address what works for whom and under what conditions. For example, studies must identify *under what conditions* and *why* teacher professional development can improve mathematics teaching and *if and how* improved teaching translates into improved student learning and achievement. Professional development that helps teachers follow through on effective mathematics interventions provided at earlier grade levels or that is teacher led and targets specific mathematics content and new ways to teach it holds particular promise. Professional development for implementing interventions for students with mathematics learning difficulties is also needed.

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